ANALYTICAL DEDUCTION OF THE RELATION BETWEEN THE MAXIMUM AND AVERAGE VELOCITIES IN PIPES OF SQUARE AND RECTANGULAR SHAPED SECTION

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Abstract. *In the present study takes into account an analytical deduction of the relation between the maximum and average velocities of the flow in ducts and pipes of rectangular and square shaped section, with the purpose of getting an expression to enable simple and fast calculation of the flow rate in ducts. Starting from the equation that models the flowing fields in rectangular and square pipes, expressions to calculate the maximum and average velocities of the flow are obtained; dividing them the relation between them is obtained. From the result, it may be observed that both depend on the kind of fluid, on the width of the transversal section of tube, and on the aspect ratio of the duct. In opposition, the relation between the maximum and the average velocities depend only on the aspect ratio of the duct. It's important to point out that the velocity relation increases when the section becomes more symmetrical, that is, the velocity relation increases when the section approaches an almost square shape.*

Keywords*: no circular pipe, maximum and average velocities of draining, no conventional flow.*

1. Introduction

In engineering ducts and pipes with circular and no circular section such as square, rectangular, triangular, elliptical, and other sections are used. The circular section pipes are the most widely used, followed by square and rectangular section pipes. These last two are applied in:

- Refrigeration and air-conditioning ducts;
- Plate heat exchanger, "Kays and London (1964)";
- Liquid-liquid Plate fin cross-flow heat exchanger. In this application, the use of rectangular and square pipe causes an increase of the turbulence, and the consequent heat exchange.
- Square and rectangular closed channels.

The internal flows in ducts with circular section have been widely studied, due to the easy analysis that can be made, and because it´s frequently used in the factory. On the other hand, there are few studies about ducts and pipes with no circular section. The relation between maximum and average velocities is an important parameter to be determinated in the ducts and pipes with no circular section. This parameter easily enables the obtainment of the flow rate. Measuring the maximum velocity in the center of the tube with a Pitot tube, and knowing the ratio between maximum and averague velocity, the averague velocity is determinated. the flow rate can be calculated multiplying this velocity by the area of the transversal section of the tube. Actually the flow rate in pipes with no circular section is determinated by measuring local velocities in several points of the transversal section, with further calculus of the average velocity using these local velocity values.

This work brings a study abaut the internal flow of ducts and pipes with square and rectangular sections, coming to an analytical deduction for the maximum and average velocities as well as the relation between them.

2. Flow in the rectangular pipe

In "Figure 1", the flow of a fluid in a rectangular pipe is observed, where the coordinate system coincide with the center of the tube and the only velocity component acts in the X axis direction.

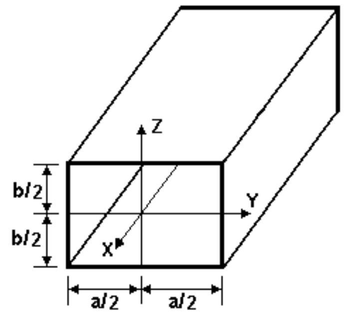

Figure 1. Rectangular tube and coordinate system

2.1 Velocity Field

General equation

In order to obtain the velocity field, starting from Navier-Stockes equation in the X-axis direction, we have:

$$\rho\left(\frac{\partial u}{dt}+u\frac{\partial u}{dx}+v\frac{\partial u}{dy}+w\frac{\partial u}{dz}\right)=\mu\left(\frac{\partial^2 u}{dx^2}+\frac{\partial^2 u}{dy^2}+\frac{\partial^2 u}{dz^2}\right)+\rho g_x-\frac{\partial P}{dx} \quad (1)$$

u : Velocity of fluid in the direction of the X axis [m/s]
v : Velocity of fluid in the direction of the Y axis [m/s]
w : Velocity of fluid in the direction of the Z axis [m/s]
ρ : Specific mass of the fluid [kg/m^3]
μ : Absolute Viscosity of the fluid [N.s/m^2]
g_x :Gravity acceleration in the direction of the X axis [m/s^2]
∂P/∂x :Linear pressure variation due to the friction in the direction of the flow [N/m^3]

Considerations:

- Laminar flow
- Stationary state
- Non-compressible fluid
- Completely developed flow (the velocity **u** does not vary in the direction of X axis.)
- The gravity only acts in the direction of Z-axis.

Particular equation

Munson *et al*. (1994)" Considering the statements above the differential equation that govern this phenomenon is obtained.

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{\mu}\frac{dP}{dx} \quad (2)$$

Boundary conditions

Considering that the variation of the velocity is two-dimensional, we can express the velocity as: u (y, z), and considering that the fluid velocity at the walls is equal to zero it is possible to write:

$$u(a/2,z)=0 \quad (3)$$

$$u(y,b/2)=0 \quad (4)$$

Once the velocity field in the center of the duct is symmetric, it is observed that:

$$\frac{\partial u(0,z)}{\partial y} = 0 \tag{5}$$

$$\frac{\partial u(y,0)}{\partial z} = 0 \tag{6}$$

Velocity Equation

In order to find the velocity profiles, it is necessary to solve only "Eq. (2)", simultaneously with the boundary conditions expressed in the "Eqs. (3)", (4)", and (6)". "Timoshenko and Goodier (1970)" have developed a solution of an analogous equations system for the torsion in rectangular bars; later, Knudsen and Katz (1958)" adapted that equation for flow of fluids. This last one is expressed as:

$$u = \frac{-4a^2}{\pi^3 \mu} \cdot \frac{\partial P}{\partial x} \cdot \sum_{n=1,3,5} \cdot \frac{1}{n^3} (-1)^{(n-1)/2} \left[1 - \frac{Cosh(\mathrm{n\pi z / a})}{Cosh(\mathrm{n\pi b / 2a})} \right] Cos(\mathrm{n\pi y / a}) \tag{7}$$

Defining the aspect ratio of the pipe **α,** as the quotient between height (b) and the width (a) of the pipe, we can write:

$$\alpha = b/a \tag{8}$$

The "Equation (7)" becomes:

$$u = \frac{-4a^2}{\pi^3 \mu} \cdot \frac{\partial P}{\partial x} \cdot \sum_{n=1,3,5} \cdot \frac{1}{n^3} (-1)^{(n-1)/2} \left[1 - \frac{Cosh(\mathrm{n.\pi.z / a})}{Cosh(\mathrm{n.\pi.\alpha / 2})} \right] Cos(\mathrm{n\pi y / a}) \tag{9}$$

2.2. Maximum velocity

The maximum velocity in ducts with rectangular and square sections is in the center of the transversal section of the tube, as observed in the paper of "Basulto and Carrocci (2000)".

$$y=0 \;\; e \;\; z=0 \;\Rightarrow\; u = u_{max}$$

Applying this condition to the "Eq. (9)" we have:

$$u_{max} = \frac{-4a^2}{\pi^3 \mu} \cdot \frac{\partial P}{\partial x} \cdot \sum_{n=1,3,5} \cdot \frac{1}{n^3} (-1)^{(n-1)/2} \left[1 - \frac{Cosh(0)}{Cosh(\mathrm{n.\pi.\alpha / 2})} \right] Cos(0) \tag{10}$$

$$u_{max} = \frac{-4a^2}{\pi^3 \mu} \cdot \frac{\partial P}{\partial x} \cdot \sum_{n=1,3,5} \cdot \frac{1}{n^3} (-1)^{(n-1)/2} \left[1 - Sec.h(\mathrm{n.\pi.\alpha / 2}) \right] \tag{11}$$

"Equation (11)" shows that the maximum velocity $\mathbf{u_{max}}$ depends on the width of the transversal section of tube **a**, the fluid viscosity μ, the pressure variation in duct **∂P/∂x** and the aspect ratio of tube **α**.

"Equation (11)" also shows that consirering a defined value of **α,** the sum depends only of **n**, and for a value of $n \geq 7$ the sum can be truncated without affecting the result significantly. Hence, the term inside of the sum will have a constant value. Then the "Eq. (11)" can be written as:

$$u_{max} = \frac{-4a^2}{\pi^3 \mu} \cdot \frac{\partial P}{\partial x} C(\alpha) \tag{12}$$

C(α) is a coefficient that only depend of α.

In “Table 1” the values of $C(\alpha)$ as a function of α, in a great interval of value, are presented.

Table 1. Variation of coefficient $C(\alpha)$

α	$C(\alpha)$
0,2	0,03802
0,4	0,14801
0,6	0,29587
0,8	0,44334
1	0,57035
1,2	0,67166
1,4	0,74922
1,6	0,80732
1,8	0,85032
2	0,88197

In “Table 1 and “Figure 2” it is observed that the coefficient $C(\alpha)$ increases as α rises.

2.3. Average velocity

It is known that the average velocity of a flow can be calculated, by integrating the velocity equation in the transversal section area of the flow. It is:

$$\overline{U} = \frac{1}{A}.\iint_A u.dA \qquad (13)$$

Replacing “Eq. (9)” in “Eq. (13)” , we have:

$$\overline{U} = \frac{1}{A}.\frac{-4a^2}{\pi^3\mu}.\frac{\partial P}{\partial x}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}.\sum_{n=1,3,5}.\frac{1}{n^3}(-1)^{(n-1)/2}\left[1-\frac{Cosh(n.\pi.z/a)}{Cosh(n.\pi.\alpha/2)}\right]Cos(n\pi y/a)dy.dz \qquad (14)$$

Integrating and applying algebraic simplifications, we have:

$$\overline{U} = \frac{1}{A}.\frac{-4a^2}{\pi^3\mu}.\frac{\partial P}{\partial x}.\sum_{n=1,3,5}.\frac{1}{n^3}(-1)^{(n-1)/2}\left[\left(\frac{a}{n.\pi}\right)(2.Sen(n\pi/2))\right]\left[b-\frac{a}{n.\pi}.\frac{2Senh(n.\pi.\alpha/2)}{Cosh(n.\pi.\alpha/2)}\right] \qquad (15)$$

It is known that $\alpha = b/a$, and the area of tube is: $A = b.a$, then, the “Eq. (15)” becomes:

$$\overline{U} = \frac{-4a^2}{\pi^3\mu}.\frac{\partial P}{\partial x}.\sum_{n=1,3,5}.\frac{1}{n^3}(-1)^{(n-1)/2}\left(\frac{Sen(n\pi/2)}{(n.\pi/2)}\right)\left[1-\frac{Tgh(n.\pi.\alpha/2)}{(n.\pi.\alpha/2)}\right] \qquad (16)$$

In “Equation (16)” it can be observed that average velocity U depends on the width (**a**) on the transversal section of the tube**,** on the fluid viscosity **μ,** on the pressure variation **$\partial P/\partial x$**, and on the aspect ratio **α**.

Similarly, in the case of maximum velocity, the term inside of the sum of the “Eq. (16)” can be truncated from $n \geq 7$, then, the average velocity can be expressed as below:

$$\overline{U} = \frac{-4a^2}{\pi^3\mu}.\frac{\partial P}{\partial x}D(\alpha) \qquad (17)$$

$D(\alpha)$ is a coefficient that depends only of α.

In “Table 2”, the values of D(α) in function of α are presented.

Table 2. Variation of coefficient D(α)

α	D(α)
0,2	0,02255
0,4	0,07728
0,6	0,1455
0,8	0,21295
1	0,27238
1,2	0,32188
1,4	0,36216
1,6	0,39478
1,8	0,42132
2	0,44312

“Table 2” shows that coefficient D(α) increases as α increases. The values of D(α) are lower than the correspondent values of C(α), as observed in the “Fig. 2”. In this figure it is also observed that the increase of parameter C(α) with the rise of α is faster than the increase of parameter D(α).

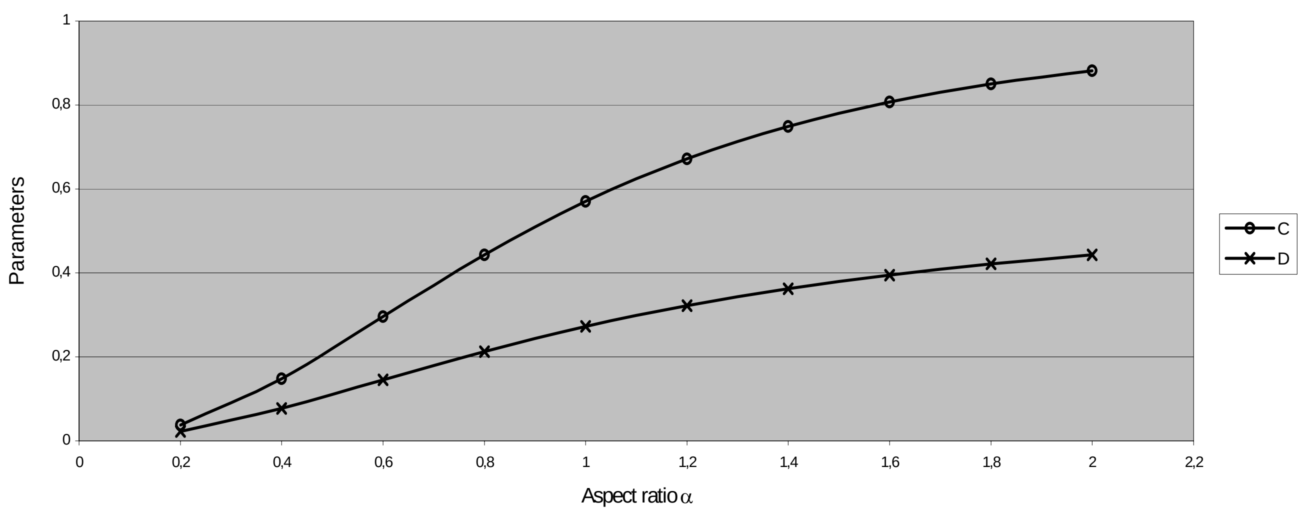

Figure 2. Variation of the parameters C and D versus aspect ratio

2.4. Relation between maximum and media velocities

The relation between the maximum and average velocities in the rectangular tube can be calculated dividing the “Eq.(12)” by “Eq. (17)”.

$$\frac{u_{max}}{\overline{U}} = \frac{C(\alpha)}{D(\alpha)} \tag{18}$$

From the “Equation (18)” can be observed that the relation between maximum and average velocities depends only on the aspect ratio of tube α (ratio between the sides of the transversal section), that relation doesn’t dependent on fluid type and on the particular measures of the pipe.

The parameters C(α) and D(α) are know for a specific value of α. Consequently, the value of the velocity relation is also known. In “Table 3” and in “Figure 3” these values are presented.

Table 3. Variation of the velocities relation u_{max}/U

α	u_{max}/U
0,2	1,68643
0,4	1,91523
0,6	2,03349
0,8	2,08188
1	2,09393
1,2	2,08667
1,4	2,06876
1,6	2,045
1,8	2,01826
2	1,99036

From “Table 3” and “Figure 3” it is possible to observe that the maximum value of the velocity relation u_{max}/U occurs when α=1, that is, when the sides of the tube are equals, or when the transversal section of the tube is a square.

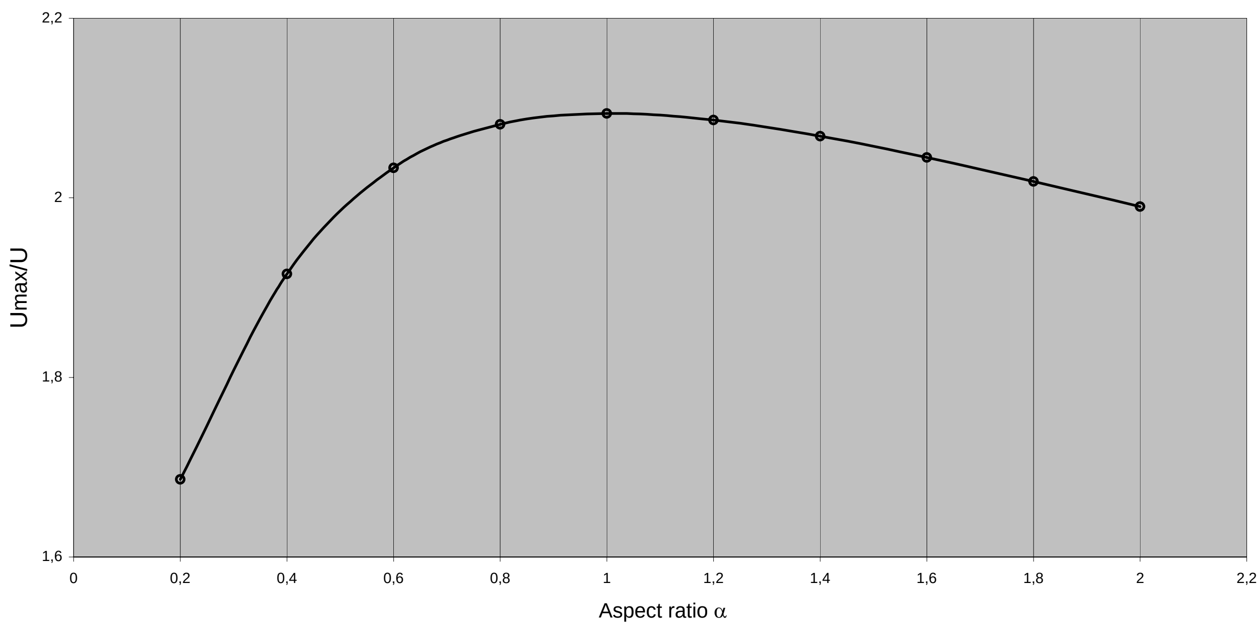

Figure 3. Variation of the relation between maximum and average velocities versus aspect ratio

3. Conclusions

- The deduction and obtainment of an analytical relationship that allows to evaluate the flow rate in rectangular pipes and ducts is the main contribution of this paper.
- The relation between maximum and average velocities depends only on the aspect ratio. This relation is independent on fluid properties, on the variation of the linear pressure in the pipe or on the size of the transversal section of the tube. For this last one, it doesnt matter if the transversal section is great or low. Tubes with the same aspect ratio have the same relation between maximum and average velocities.
- The relation between the maximum and average velocities assumes its biggest value when the transversal section of the tube is a square (more symmetric). It can be explained because the velocity ratio is greater in the laminar flow, and lower when the flow becomes turbulent.
- The relation between maximum and average velocities diminishes considerably when the transversal section of the rectangular tube is very different from a square.
- The maximum and average velocities are influenced by the viscosity of the fluid, the variation of the linear pressure, the measurements of the transversal section of the tube and by the aspect ratio.

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