PERFORMANCE ANALYSIS BETWEEN DIFFERENTS LINEARIZATIONS SCHEMES IN THE DISCRETIZATION OF MOVEMENT EQUATIONS WITH VORONOI DIAGRAMS

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Abstract. *This paper presents the results obtained with Taylor-series linearization and Separation linearization, with u and v velocities co-localized stored and evaluated through UDS (UpWind Difference Scheme) and CDS (Central Difference Scheme) interpolation schemes for discretization equations of Navier-Stokes on the Finite Volume Method with non-structured mesh generated by Voronoi Diagram. The proposal of this work is the computational cost optimization involved on the linear equations system solution generated from discretization. In this way, the analysis approached for this work is restricted for Taylor-series linearization with UDS and CDS, and Separation with UDS. The Conjugated Gradient method was used to solve the linear system solution equation from discretization. To validate the numeric results it was used the laminar incompressible flow on a squared cavity of finite depth with a sliding superior wall with steady velocity, like a treadmill. Based on the work results, the Taylor-series linearization with u and v velocities evaluated with CDS presented better performance than Taylor-series linearization with UDS and Separation with UDS.*

***Keywords**: Taylor-series linearization, Separation Linearization, Central Difference Scheme, UpWind Difference Scheme, Navier-Stokes Equations.*

1. Introduction

Once the work of discretization of Navier-Stokes is done, the system equations has non linear terms of second order which needs a linearization process so it can be resolved as a linear equations system. The type of linearization may affect the equation system convergence generated from the discretization, as well as the adopted interpolation scheme to evaluate the u and v on the control volume (CV).

Usually, some interpolation schemes are stable, but they present low precision. There are others, however, which have god precision, but are oscillatory. So, in this context, this work presents the results obtained with the Taylor-series linearization and Separation linearization, with u and v velocities with colocated stored and evaluated through UDS interpolation scheme (UpWind Difference Scheme) and CDS (Central Difference Scheme), for discretization of Navier-Stokes equations on the Finite Volume Method (FVM) with non-structured mesh generated by Voronoi Diagram (VD). The proposal of this work is the computational cost optimization involved on the solution of linear equations system generated from discretization. In this way, the analysis approached for this work is restricted for Taylor-series linearization with UDS and CDS and Separation with UDS.

2. Equations Linearization of Navier-Stokes

The main objective of this section is to present the way that velocities u and v was evaluated (with colocated stored) through the UDS and CDS interpolation schemes on the discretization of Navier-Stokes equations on the FVM with non-structured mesh generated by VD. Before this, however, it will be presented a brief introduction to the Taylor-series linearization and to the Separation linearization, with the objective to introduce this work and emphasize the use of the interpolations schemes previously summoned.

2.1. Discretization of Navier-Stokes Equations and Separation and Taylor-series linearization

According to Cardoso (1997), after the process of discretization of Navier-Stokes equation on the FVM, the following discretized equation is generated:

$$Ap_i^0(\varphi_i - \varphi_i^0) + \sum_{j=1}^{NV(i)} \left[\rho W \varphi - \Gamma^{\varphi} \frac{\partial \varphi}{\partial z} \right]_{ij} S_{ij} = S_i^{\varphi} \Delta \vartheta_i \tag{1}$$

Where:

Ap_i^0 → Nodal point i Coefficient.
ρ_i → Specific fluid mass in the nodal point i.
t → Time.
$\Delta\vartheta_i$ → VC Volume from a polygonal convex prism generated trough VD.
φ_i^0 → Generic φ variable evaluated in the previous time (t).
φ_i → Generic φ variable evaluated in the time $t+\Delta t$.
i → Generic Control Volume (VC).
j → Neighbor index for i.
NV → Number of neighbors for i.
W_{ij} → Normal velocity Component evaluated on ij face.
Γ^{φ} → Diffusion coefficient for φ.
L_{ij} → Distance between i and j.
S_{ij} → Face (edge) between two generic points i and j.
S_i^{φ} → Is the source term for φ on the VC i.
$\left.\frac{\partial\varphi}{\partial z}\right|_{ij} = \frac{\varphi_i - \varphi_j}{L_{ij}}$ → Approach for central difference of 1st order ($E_T = O^2$).

In the Equation (1) there is a nonlinearity involved in the advective-difusive term - Eq. (2) – when φ assumes the value of some velocity component ($\varphi = u$ or $\varphi = v$). Usually, the equations system generated from the discretization process are linearized, or in the discretization of non linear terms, or in the linearization of source terms and the solution of this nonlinearity are made of iterative mode. So, the term $\rho W\varphi$ must be linearized to solve the algebraic system generated from discretization as a linear equation system.

$$J_{ij} = \left[\rho W\varphi - \Gamma^{\varphi}\frac{\partial\varphi}{\partial z}\right]_{ij} \tag{2}$$

Traditionally, the term $\rho W\varphi$ value is obtained from previous iteration velocity values (showed with *) and φ is evaluated on the new iteration generating the Separation linearization - Eq. (3).

$$[\rho\, W\, \varphi]_{ij} = \rho_{ij} W_{ij}^{*}\varphi_{ij} \tag{3}$$

Alternatively, Taylor-series formulation could be applied to non linear system (Alonso, 1999) writing $\varphi_{ij} = u_{ij}$ and $\varphi_{ij} = v_{ij}$ in the advective-difusive term presented in Eq. (2):

$$G(u_{ij}, v_{ij}) \cong G(u_{ij}^{*}, v_{ij}^{*}) + \left.\frac{\partial G}{\partial u_{ij}}\right|_{u_{ij}^{*}, v_{ij}^{*}}(u_{ij} - u_{ij}^{*}) + \left.\frac{\partial G}{\partial v_{ij}}\right|_{u_{ij}^{*}, v_{ij}^{*}}(v_{ij} - v_{ij}^{*}) \tag{4}$$

$$H(u_{ij}, v_{ij}) = H(u_{ij}^{*}, v_{ij}^{*}) + \left.\frac{\partial H}{\partial u_{ij}}\right|_{u_{ij}^{*}, v_{ij}^{*}}(u_{ij} - u_{ij}^{*}) + \left.\frac{\partial H}{\partial v_{ij}}\right|_{u_{ij}^{*}, v_{ij}^{*}}(v_{ij} - v_{ij}^{*}) \tag{5}$$

Thus there are two alternatives to solve the problem of the non-linearity involved. If in one hand the Separation linearization has been traditionally used and presents good results, on the other hand the linearization using Taylor-series formulation has a tendency to be consistent because presents less oscillations and conducts a faster convergence process – normally Newton's method produces quadratically convergent approximations (Faires and Burden, 1993). Thus, once the problem of non- linearity involved on the discretization through linearization scheme is solved, it is possible to apply some interpolation scheme with the objective to evaluate the u and v velocities. This is showed in the sequence.

2.2. *u* and *v* Velocities Evaluation Through UDS e CDS Interpolation Schemes

Adopting a UDS interpolation scheme to Taylor-series linearization (Alonso, 1999), *u* and *v* could be evaluated in an *ik* interface – Fig. (1) – generating the followed linear interpolation for u_{ik} :

$$u_{ik} = u_i \; \delta_{ik} + u_{r(k)} \; \delta_{ik} \tag{6}$$

Where:

$$\Psi\left(\delta_{ik}\right) = \begin{cases} 1, if \; \delta_{ik} > 0 \\ 0, if \; \delta_{ik} \leq 0 \end{cases} \tag{7}$$

If $W_{ik}^{*} > 0$ then $\delta_{ik} = 1$ and $\delta_{ik} = 0$, so $u_{ik} = u_i$. If $W_{ik}^{*} < 0$ then $\delta_{ik} = 0$ and $\delta_{ik} = 1$, so $u_{ik} = u_{r(k)}$.

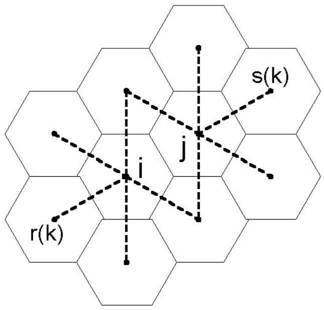

Figure 1. *i* and *j* and their respective neighborhoods *r*(*k*) e *s*(*k*).

The generic interpolation to v_{ik} could be obtained in an analog way that was showed to u_{ik} . Therefore:

$$v_{ik} = v_i \; \delta_{ik}^{+} + v_{r(k)} \; \delta_{ik}^{-} \tag{8}$$

Considering the Taylor-series linearization, *u* and *v* could be evaluated using the CDS interpolation (Alonso, 1999):

$$u_{ik} = \frac{\left(u_i + u_{r(k)}\right)}{2} \tag{9}$$

$$v_{ik} = \frac{\left(v_i + v_{r(k)}\right)}{2} \tag{10}$$

Finally, the Separation linearization with UDS may be found in Cardoso (1997) or Mariani (1997).

3. Results and Conclusions

The test used to evaluate the linearization with their respective interpolations proposed in this work was the laminar incompressible flow on a squared finite depth cavity with a sliding superior wall (Lid-driven square cavity) with steady velocity, like a treadmill, Fig. (2). Inferior and lateral walls are impermeable. The boundary conditions used were:

- Right lateral wall: x = L, $y = 0$, $u = v = 0$.
- Left lateral wall: $x = 0$, $u = v = 0$.
- Inferior wall: $y = 0$, $u = v = 0$.
- Superior wall: $y =$ L, $u = 1$, $v = 0$.
- Width of the mesh: $L = 32.10$ (value chosen in function of the mesh).

A computational program was used to solve the proposed problem (Cardoso, 1997), in which there were implemented the linearizations and their respective interpolations compared in this work, simulating the stationary fluid flow, although the implemented program permits the temporal discretization too.

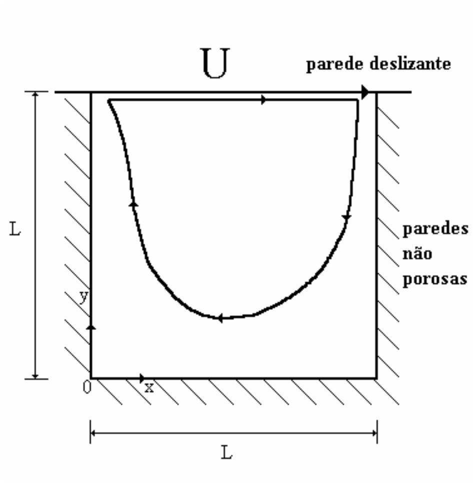

Figure 2. Lid-driven square cavity.

For u and v velocities it was adopted the 0.7 relaxation. For the pressure it was used 0.5. The internal repetition number for u and v velocities was two, while for pressure was three. It was adopted $u = v = 0.5$ for initials velocity conditions and for pressure $P = 0$. The solution to the linear equations system generated from discretization was obtained through Conjugated Gradient method (Mariani, 1997) (Golub, 1989) and the pressure gradient used was the one proposed by Taniguchi *et al.* (1991a) and Taniguchi and Kobayashi (1991b).

For test cases it was used a mesh of hexagonal volumes with 6400 nodal points (velocity collected on point 35) generated through VD (mesh generated for the generator of VD of Maliska Jr. (1994)). For the boundary, the program allowed to use a unique neighbor for all nodal points connected to a wall; which was very useful for the proposal system solution. It's important to point out that the boundary conditions could be inserted in the discretized equations, but this weren't made because more than a condition of boundary in each side of the mesh wasn't needed. The results were collected with Reynolds numbers 100 and 1000, and calculated by:

$$Re = \rho \frac{U.L}{\mu} \tag{11}$$

Where U is the superior wall movement speed ($U = 1$), L is the cavity width ($L = 32.10$), ρ is the specific fluid mass ($\rho = 1$) and μ is the dynamic fluid viscosity (it varies as the desired Reynolds number).

As convergence criteria (according to the desired precision, $\xi = 1.\ 10^{-5}$) it was used:

$$error_i^k = \sum_{j=1}^{NV(i)} \left| F_{ij}^k \right| < \xi \tag{12}$$

The results obtained with linearization and their respective interpolations proposed in this work were compared with themselves and with the results obtained by Ghia *et. al.* (1982).

Finally, the figures shown on the next subsections present the graphics with the velocity u and v profiles for Reynolds 100 and 1000 throughout a central vertical line ($x = L / 2$) and horizontal central line ($y = L / 2$) (located in the geometric center of the squared cavity) for the velocities in directions x and y, respectively, on the used mesh.

3.1. Results Obtained with Reynolds 100

To the graphic presented in left side in Fig. 3 the results showed up that Separation and Taylor-series linearization, both with UDS, were very close and sometimes overlapped to the results obtained by Ghia *et. al.* (1982). On the other hand, to the graphic presented in right side in Fig. 3 the results showed up that the obtained results with Taylor-series linearization with CDS moved away slightly from the obtained results with Separation linearization with UDS and from the results obtained by Ghia *et. al.* (1982).

To the graphic presented in left side in Fig. 4 the results showed up that Separation and Taylor-series linearization, both with UDS, overlapped each other completely, and although they did not present results relatively close to the results obtained by Ghia *et. al.* (1982). On the other hand, to the graphic presented in right side in Fig. 4 the results showed up that the obtained results with Taylor-series linearization with CDS moved away from the obtained results with the Separation with UDS which on its turn moved away from the results obtained by Ghia *et. al.* (1982).

According to Tab. 1, it was possible to conclude that Taylor-series linearization with CDS presented best performance when considerate the number of iterations and the CPU time needed to convergence of system equation.

Table 1. Taylor-series linearization versus Separation Linearization - *Re* = 100.

Linearization type	Number of Iterations	CPU time (minutes)	Velocity[35].*u*
Separation com UDS	1654	24,05	-0.004896
Taylor-series linearization with UDS	1309	18,28	-0.004895
Taylor-series linearization with CDS	1235	16,18	-0.005439

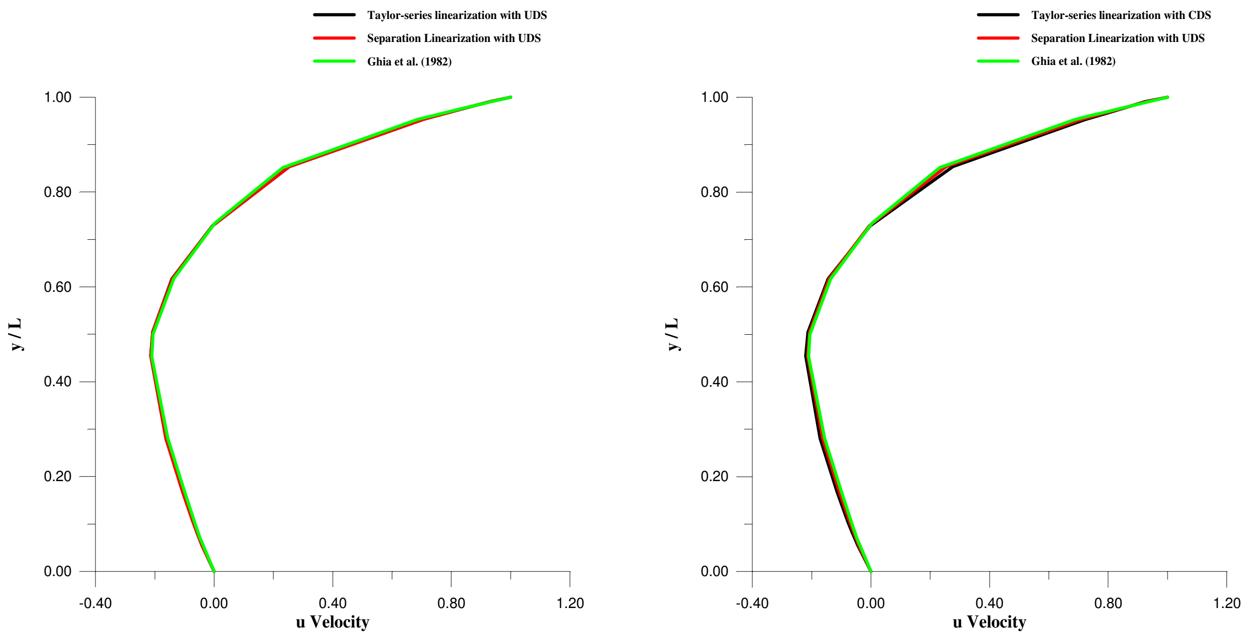

Figure 3. Velocity *u* throughout a central vertical line with *R*e = 100 and Taylor-series linearization with UDS and CDS, Separation Linearization with UDS and Ghia *et. al.* (1982).

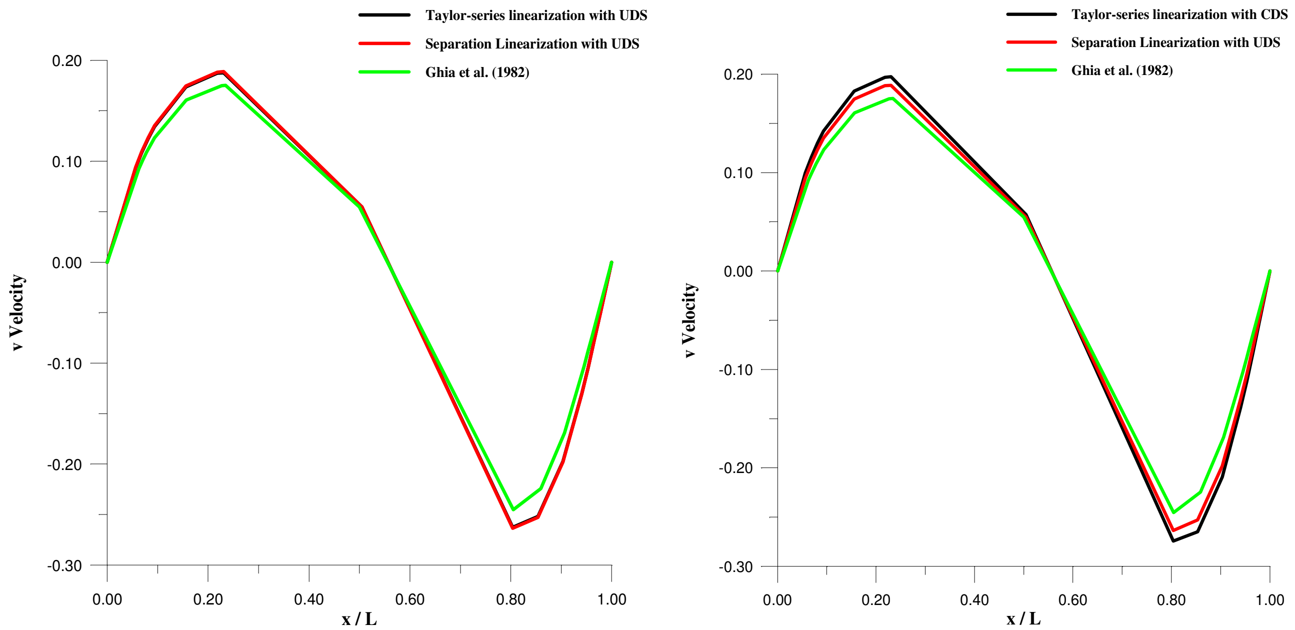

Figure 4. Velocity *v* throughout a central horizontal line with *R*e = 100 and Taylor-series linearization with UDS and CDS, Separation Linearization with UDS and Ghia *et. al.* (1982).

3.2. Obtained Results with Reynolds 1000

The graphic presented in left side in Fig. 5 showed up that the obtained results with Separation and Taylor-series linearization, both with UDS, overlapped each other completely and sometimes overlapped the results obtained by Ghia *et. al.* (1982). On the other hand, the graphic presented in right side in Fig. 5 showed up that the obtained results with Taylor-series linearization with CDS moved away slightly from the obtained results with Separation linearization with UDS which in its turn move away slightly from the results obtained by Ghia *et. al.* (1982).

The graphic presented in left side in Fig. 6 showed up that the obtained results with Separation and Taylor-series linearization, both with UDS, overlapped each other completely and sometimes overlapped the results obtained by Ghia *et. al.* (1982). On the other hand, the graphic presented in right side in Fig. 6 showed up that the obtained results with Taylor-series linearization with CDS moved away from the results obtained with Separation with UDS which in its turn moved away from the results obtained by Ghia *et. al.* (1982).

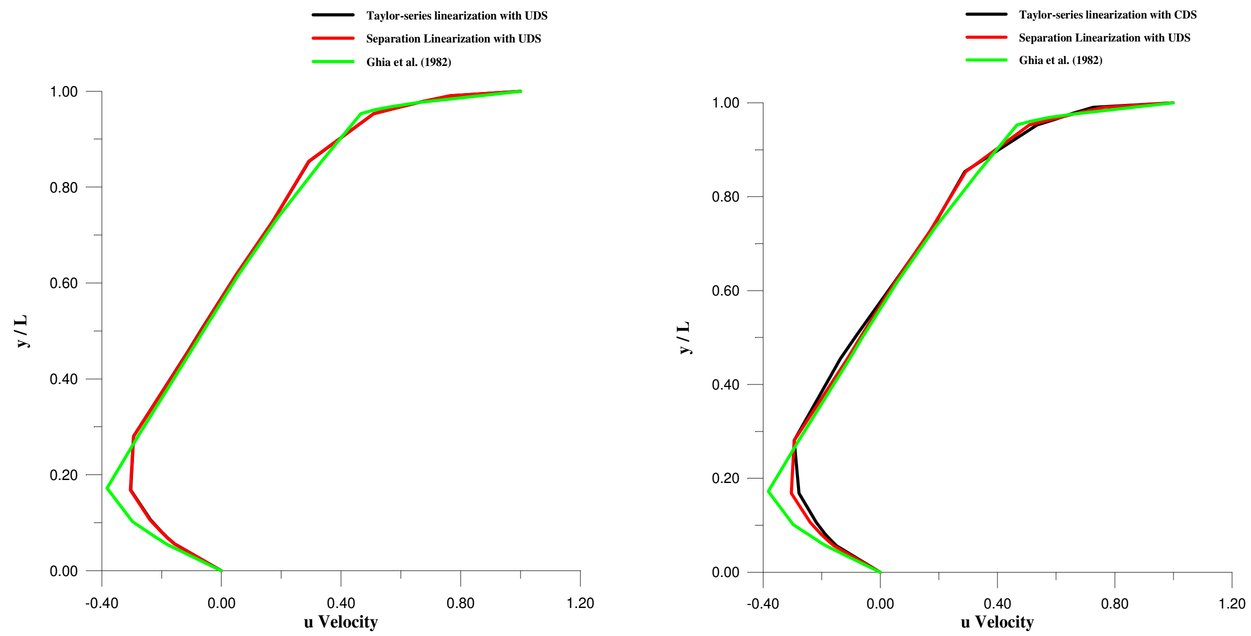

Figure 5. Velocity u throughout a central vertical line with Re = 1000 and Taylor-series linearization with UDS and CDS, Separation Linearization with UDS and Ghia *et. al.* (1982).

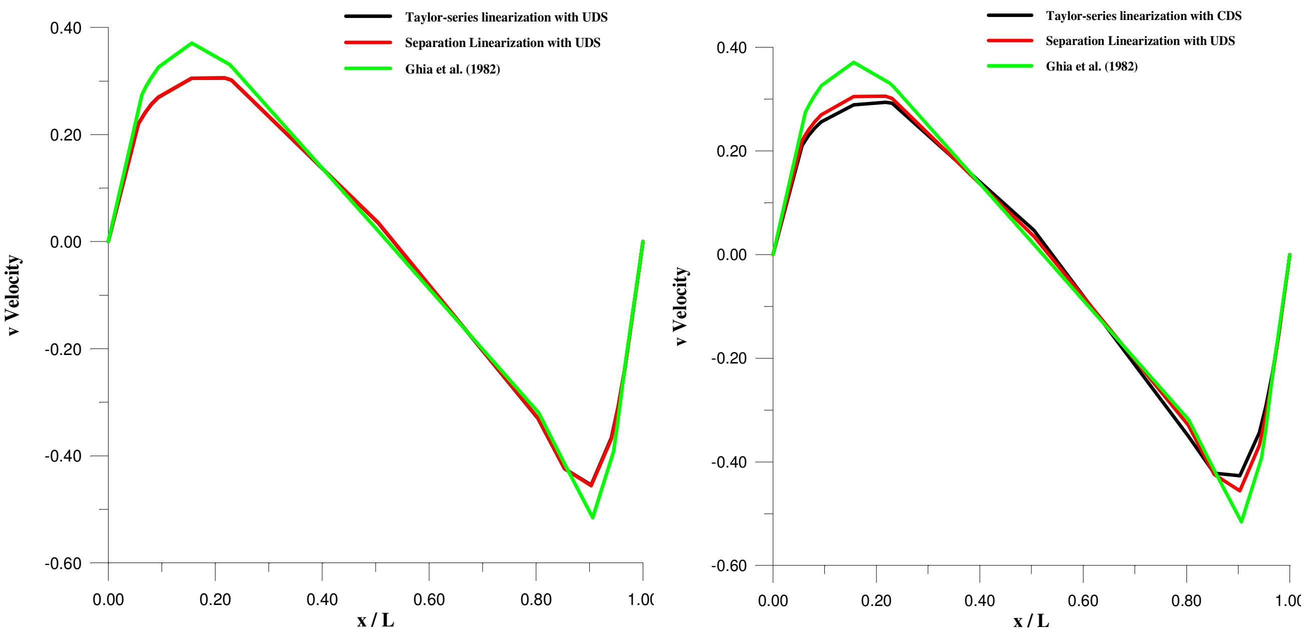

Figure 6. Velocity v throughout a central horizontal line with Re = 1000 and Taylor-series linearization with UDS and CDS, Separation Linearization with UDS and Ghia *et. al.* (1982).

According to Tab. 2, it was possible to conclude that Separation linearization with UDS presented best performance in CPU time, while Taylor-series linearization with CDS presented best results on number of iterations.

Table 2. Taylor-series linearization versus Separation Linearization - *Re* = 1000.

Linearization Type	Number of Iterations	CPU Time (minutes)	Velocity [35].*u*
Separation with UDS	2325	32,17	-0.021100
Taylor-series linearization with UDS	2516	37,27	-0.020983
Taylor-series linearization with CDS	2247	32,62	-0.023952

4. Final Considerations

During the test cases it was observed that the system equations convergence velocity through Taylor-series linearization was dependent from the used relaxation factors and the number of internal repetition to velocities u and v. It must be considerated that the work presented on this paper has not as purpose to indicate the best values for this coefficients, since it would spend a great number of tests, even considering the nonlinearity involved and the correct relaxation factor to speed up or to slow down dependent variable (velocities u and v).

Although it had been presented results for Reynolds's number 100 and 1000 only, tests have been made to Reynolds equal 10000, but the process of convergence to the equations system wasn't obtained. Additional tests must be done to obtain more information about this event.

To the velocities u and v profiles presented on the previous sections, it is important to point out that the mesh used by Ghia *et. al.* (1982) has about 16641 nodal points, while the mesh used on this work has 6400 nodal points disposed in hexagonal volumes.

Finally, it must be emphasized that the present work used segregated solution for equations system because the Conjugate Gradient method with the way it was implemented resolves the u and v velocities separately.

5. References

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5. Responsibility notice

The authors Ewerton Eyre de Morais Alonso and Sérgio Peters are the only responsible for the printed material included in this paper.