A CRITICAL EVALUATION OF THREE MODELS OF EXTERNAL BALLISTICS

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Abstract. *In the present work, performance analysis of three different exterior ballistics models for predicting projectile trajectories is examined. The models selected are: point-mass, modified point-mass and 6-DOF. The objective of the analysis is to verify which of these models provides the most accurate predictions for both range and maximum height of a trajectory, given its initial conditions, projectile characteristics and associated atmospheric effects. The obtained results are compared with experimental data, which leads to the conclusion that 6-DOF is the best among the three approaches, although it requires a larger time of computational processing.*

Key-words: *Exterior Ballistics Models; Point-Mass; Modified Point-Mass; 6-DOF; Six-Degrees-Of-Freedom.*

1. Introduction

The accurate prediction of ballistic trajectories constitutes a considerably complex problem, due to theoretical difficulty in modeling aerodynamic forces and moments that act upon the body during its ballistic flight across the atmosphere. These difficulties increase mostly with the increase of speed and with the increase of release angle, that is, for greater ranges or for higher trajectories. In these cases, aerodynamic effects are not negligible in comparison with gravity force and the trajectory is neither flat nor parabolic. In fact, the trajectory is asymmetric and both the condition of the atmosphere and the shape of the body have a significant effect on its tracing [McCoy, 1999].

Knowledge of the ballistic trajectory of aerodynamic bodies is of fundamental importance, because, combined to knowledge of the topography of the region where the release shall be accomplished, it is possible to determine whether the referred body is going to hit any natural accident before reaching its goal.

In this context, aiming to investigate the different models' performance in predicting the ballistics trajectories of bodies stabilized by spinning (also called projectiles), three models were selected [McCoy, 1999]. The first of them, called Point-Mass, has only three degrees of freedom for the projectile's motion, considering only its translation. The second model, called Modified Point-Mass, adds the spinning motion to Point-Mass's translation, describing the trajectory with four degrees of freedom. The last model, called Six-Degrees-Of-Freedom (6-DOF), is based on a minimal number of simplifications and its equations are considerably more complex than those of the previous models. Exactly like its name, 6-DOF computes the translation and rotation motion summing six degrees of freedom.

At this work, the ballistic trajectories predicted by each selected model were compared with experimental data in order to evaluate performance as well as computational efficiency associated to each of the approaches.

2. Physical Phenomenon

The problem in study is the simulation of the three-dimensional trajectory described by a projectile stabilized by rotation, from the moment of its launching, when its position and velocity are known, until it reaches again the horizontal plan determined by launching position.

With this purpose, it becomes necessary to model mathematically the dynamics of the projectile, considering the varied effect of forces and moments that act on the body. In synthesis, these effects include efforts of gravitational and aerodynamic origins, which vary according to position and velocity of the projectile. Applying to such an effect the principles of Classic Mechanics, we are left with a system of six non-linear differential equations, whose solution represents the position of the projectile as a function of time.

A global vision of the physical phenomenon to be mathematically modeled allows us to enumerate and classify the most important parameters that influence the flight of the projectile. These parameters are: the own characteristics of the projectile, such as geometry (Fig. 1), inertia and aerodynamic coefficients; the environment in which the release is inserted, characterized by air density, wind velocity, Coriolis' acceleration, gravity and sound speed; and the initial conditions of launching, in other words, muzzle velocity and initial position of the projectile. In the particular case of a numerical simulation, the mathematical model and the numerical computational method used evidently influence the predicted trajectory as well.

3. Mathematical models

All three models studied here adopt as hypotheses the following [McCoy, 1999]: rigid, non-finned and axially symmetric projectile (Fig. 1) and constant inertia, without loss of mass or alteration of projectile's mass distribution during its flight. Moreover, wind speed was considered constant for all trajectories and effects caused by variation of air humidity have been neglected. However, the effect of Earth's curvature was taken in account because of its important impact on great range launchings.

Figure 1 – Unit vector along the projectile's rotational axis of symmetry.

Fig. 2 shows the reference system adopted for measuring all coordinates, with origin in the launching position, *xz*-plane parallel to the ground and y-axis in vertical direction.

The aerodynamic forces considered in the modeling are [McCoy, 1999]: Drag Force, Lift Force, Magnus Force and Pitch Damping Force. The aerodynamic moments considered are [McCoy, 1999]: Spin Damping Moment, Overturning Moment, Magnus Moment and Pitch Damping Moment. Any other aerodynamic effect is neglected due to its small influence upon the final shape of the trajectory [McCoy, 1999].

For the mathematical characterization of the atmosphere, it was used the standard atmosphere supplied by the International Civil Aviation Organization, usually known as ICAO atmosphere [McCoy, 1999].

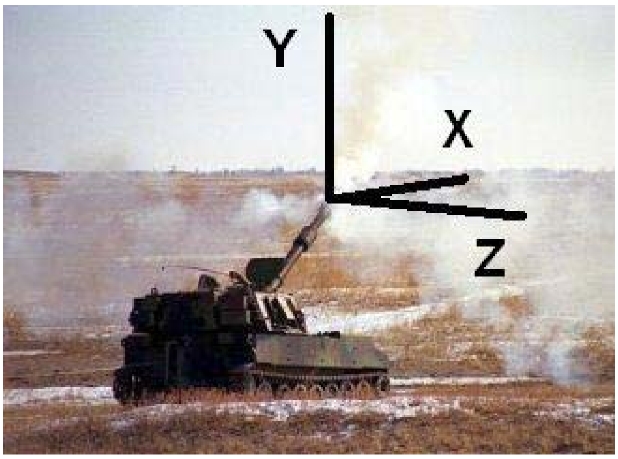

Figure 2 – Reference system.

3.1 Point-Mass Model

Point-Mass modeling consists of considering all the mass of the projectile concentrated on a single point, which coincides with its center of mass, all rotational effects being, therefore, negligible. As a consequence of this approach, aerodynamic effects are neglected– except for Drag Force and Coriolis's acceleration - against gravitational acceleration [McCoy, 1999]. In result to the hypothesis made, the movement of the projectile is described by three degrees of freedom corresponding to the position coordinates (x, y, z).

The equations that govern the dynamics are given by [McCoy, 1999]:

$$\frac{d\vec{V}}{dt} = -\frac{\rho S C_D}{2m} \cdot \left|\vec{V} - \vec{W}\right| \cdot \left(\vec{V} - \vec{W}\right) + \vec{g} \tag{1}$$

where t is time, $\vec{V}$ is the projectile´s linear velocity vector, m is the projectile's mass, ρ is local air density, S is the projectile's reference area, C_D is the dimensionless drag coefficient, $\vec{W}$ is the wind velocity vector and $\vec{g}$ is the acceleration vector due gravity.

3.2 Modified Point-Mass Model

The Modified Point-Mass Model considers the projectile's rolling motion around its longitudinal axis of symmetry, called spinning motion [McCoy, 1999]. Thus, this model presents four degrees of freedom: three translational coordinates for describing position and one for angular speed.

An important effect that must be commented is the angle of attack (or yaw). It is the angle between the projectile's velocity vector (measured relatively to the atmosphere) and the unit vector oriented in the same direction as the projectile's axis of symmetry. In the modified point-mass model, this angle is incorporated into the mathematical equations as an estimated average value (yaw of repose) that influences the calculation of Lift and Magnus forces, both of aerodynamic nature [McCoy, 1999].

The basic equations of this model are:

$$\vec{\alpha}_R = \frac{2 I_x p \left[\vec{g} \times (\vec{V} - \vec{W})\right]}{\rho S d \left|\vec{V} - \vec{W}\right|^4 C_{M_\alpha}} \tag{2}$$

$$\vec{\Lambda} = 2\Omega \begin{bmatrix} \left[-\left(V_z - W_z\right)\cos L \,\text{sen}\, AZ - V_y \,\text{sen}\, L\right]\hat{u}_x + \\ + \left[\left(V_x - W_x\right)\cos L \,\text{sen}\, AZ + \left(V_y - W_y\right)\cos L \cos AZ\right]\hat{u}_y + \\ + \left[\left(V_x - W_x\right)\text{sen}\, L - \left(V_z - W_z\right)\cos L \cos AZ\right]\hat{u}_z \end{bmatrix} \tag{3}$$

$$\frac{d\vec{V}}{dt} = \vec{g} - \frac{\rho S C_D}{2m} \cdot \left|\vec{V} - \vec{W}\right| \cdot \left(\vec{V} - \vec{W}\right) + \frac{\rho S C_{L_\alpha}}{2m} \cdot \left|\vec{V} - \vec{W}\right|^2 \vec{\alpha}_R + + \frac{\rho S d C_{N_{p\alpha}}}{2m} \cdot p \left[\left(\vec{V} - \vec{W}\right) \times \vec{\alpha}_R\right] + \vec{\Lambda} \tag{4}$$

$$\frac{dp}{dt} = \frac{\rho S p d^2 C_{lp}}{2 I_x} \left|\vec{V} - \vec{W}\right| \tag{5}$$

where $\vec{\alpha}_R$ is the vector that represents the yaw of repose, I_x is the projectile's axial moment of inertia, p is the rolling speed, d is the projectile's reference diameter, $C_{M\alpha}$ is the Overturning Moment coefficient, $\vec{\Lambda}$ is the Coriolis's acceleration vector, Ω is the angular velocity of theEarth around its polar axis, L is the latitude of the firing site, AZ is the azimuth of fire, measured clockwise from North, $\hat{u}_i$ is the unit vector along the x-axis, C_{L_α} is the Lift Force coefficitent, $C_{N_{p\alpha}}$ is the Magnus Force coefficient and C_{lp} is the Spin Damping Moment coefficient.

3.3 Six-Degrees-Of-Freedom Model

Finally this model presents six degrees of freedom (three components of the linear velocity vector and three components of the angular velocity vector). The six ordinary differential equations that represent the model can be written as [McCoy, 1999]:

$$\begin{aligned}&\vec{g}+\cdot\vec{\Lambda}-\frac{\rho S C_D}{2m}\cdot\left|\vec{V}-\vec{W}\right|\cdot\left(\vec{V}-\vec{W}\right)+\frac{\rho S C_{L\alpha}}{2m}\left[\left(\vec{V}-\vec{W}\right)\times\left(\vec{x}\times\left(\vec{V}-\vec{W}\right)\right)\right]+\\&+\frac{\rho SpdC_{Np\alpha}}{2m}\left[\left(\vec{V}-\vec{W}\right)\times\vec{x}\right]+\frac{\rho Sd}{2m}\left(C_{N\alpha}+C_{Nq}\right)\left|\vec{V}-\vec{W}\right|\cdot\frac{d\vec{x}}{dt}=\frac{d\vec{V}}{dt}\end{aligned} \tag{6}$$

$$\begin{aligned}&\frac{\rho Spd^2C_{lp}}{2I_y}\left|\vec{V}-\vec{W}\right|\vec{x}+\frac{\rho SdC_{M\alpha}}{2I_y}\left|\vec{V}-\vec{W}\right|\left[\left(\vec{V}-\vec{W}\right)\times\vec{x}\right]+\\&+\frac{\rho Spd^2C_{Mp\alpha}}{2I_y}\left[\vec{x}\times\left(\left(\vec{V}-\vec{W}\right)\times\vec{x}\right)\right]+\frac{\rho Sd^2}{2I_y}\left(C_{Mq}+C_{M\dot{\alpha}}\right)\left|\vec{V}-\vec{W}\right|\left(\vec{x}\times\frac{d\vec{x}}{dt}\right)=\frac{d\vec{h}}{dt}\end{aligned} \tag{7}$$

where $\vec{x}$ is the unit vector along the projectile's axis of symmetry, $C_{N\alpha}$ is the Pitch Damping Force coefficient due to the rate of change of yaw, C_{Nq} is the Pitch Damping Force coefficient due to the rate of change of the unit vector $\vec{x}$, $C_{Mp\alpha}$ is the Magnus Moment coefficient, C_{Mq} is the Pitch Damping Moment coefficient due to the rate of change of the transverse angular velocity, $C_{M\alpha}$ is the Overturning Moment coefficient and $\vec{h}$ is the projectile's angular velocity vector.

4. Numerical Solution

In order to solve numerically the equations that govern each one of the models, a computational code in language C was elaborated using a fourth order Runge-Kutta algorithm with constant integration step [Dieguez, 1992]. From projectile data, atmospheric information and initial conditions of release, the modularly encoded program solves the equations in an iterative form, determining the position coordinates of the projectile for each moment in time.

Simulations with different fixed steps of time have been carried through, so as to verify the sensitivity of the solution to the referred steps. Tests revealed independence towards time step from Dt = 0.01s.

It must be registered that the projectile's Drag Force coefficient depending on Mach number was given by the commercially available program Prodas 2000 [Arrow Tech, 2000]. The solution derived from vacuum ballistics and the results given by Prodas's models have been used to validate the cited code.

5. Analysis of the results

In order to compare the results given by computational implementations of the three models presented, it was selected a group of trajectories, characterized by two initial conditions: launch velocity and elevation angle. Following the majority of authors on the subject, elevation angles, defined as angles between the initial velocity vector and the horizontal plane, were measured in (ml) instead of degrees (°).

This particular group of trajectories served as a basis for the analysis of simulations. In every case, the projectile was assumed to be M1 105mm artillery ammunition, so that experimental reliable data could be obtained from FT 105-H-7 firing tables. For this type of projectile, aerodynamic coefficients varying with Mach number were extracted from Prodas, particularly from its modules dedicated to External Ballistics. In all cases, it was also assumed that launch velocity was equal to 464,8m/s, a typical value for M1 105mm projectiles. For modified point-mass and 6-DOF trajectories, initial roll velocity was assumed to be 1545rad/s, corresponding to the value of launch velocity chosen [FT 105-H-7, 1971]. Wind velocities were null throughout this analysis.

The results of simulation of each one of the trajectories obtained with the models selected are displayed in Tab. 1 and 2, which compare these values to firing table data. In Tab. 1 is accomplished the comparison of ranges, while in the

Tab. 2 the maximum height of the trajectory is compared. The comparison is carried out in two distinct ways, considering absolute and percent errors.

Table 1 – Comparison of the maximum range results.

Elevation angle θ (ml)	Firing table range (m)	Implemented models								
		Point-mass			Modified point-mass			6-DOF		
		Range (m)	Absolute error (m)	Relative error (%)	Range (m)	Absolute error (m)	Relative error (%)	Range (m)	Absolute error (m)	Relative error (%)
100	3171	3139	32	1,01	3151	20	0,63	3171	37	1,17
200	5289	5246	43	0,81	5266	23	0,43	5289	58	1,10
300	7032	6992	40	0,57	7017	15	0,21	7032	63	0,90
400	8469	8439	30	0,35	8468	1	0,01	8469	56	0,66
500	9600	9586	14	0,15	9617	17	0,18	9600	43	0,45
600	10409	10415	6	0,06	10446	37	0,36	10409	25	0,24
700	10878	10906	28	0,26	10936	118	1,09	10878	5	0,05
800	-	11037	-	-	11064	-	-	-	-	-
900	10762	10800	38	0,35	10824	62	0,58	10762	0	0
1000	10152	10207	55	0,54	10223	71	0,70	10152	10	0,10
1100	9152	9249	97	1,06	9255	103	1,13	9152	41	0,45
1200	7725	7937	212	2,74	7928	203	2,63	7725	133	1,72
1300	5617	6299	682	12,14	-	-	-	5617	535	9,52
1400	-	4380	-	-	-	-	-	-	-	
1500	-	2250	-	-	-	-	-	-	-	
1600	-	0	-	-	-	-	-	-	-	

Table 2 Comparison of the largest height results.

Elevation Angle θ (ml)	Firing Table Largest Height (m)	Implemented models								
		Point-mass			Modified point-mass			6-DOF		
		Largest height (m)	Absolute error (m)	Relative error (%)	Largest height (m)	Absolute error (m)	Relative error (%)	Largest height (m)	Absolute error (m)	Relative error (%)
100	91	89	2	2,20	90	1	1,10	90	1	1,10
200	317	312	5	1,58	313	4	1,26	313	4	1,26
300	641	635	6	0,94	637	4	0,62	637	4	0,62
400	1053	1043	10	0,95	1047	6	0,57	1046	7	0,66
500	1541	1526	15	0,97	1532	9	0,58	1530	11	0,71
600	2093	2068	25	1,20	2075	18	0,86	2073	20	0,96
700	2838	2651	187	6,59	2661	177	6,24	2657	181	6,38
800	-	3258	-	-	3270	-	-	3265	-	-
900	3721	3866	145	3,90	3880	159	4,27	3874	153	4,11
1000	4456	4455	1	0,02	4471	15	0,34	4464	8	0,18
1100	5014	5003	11	0,22	5020	6	0,12	5013	1	0,02
1200	5500	5488	12	0,22	5507	7	0,13	5498	2	0,04
1300	5735	5891	156	2,72	-	-	-	5854	119	2,07
1400	-	6193	-	-	-	-	-	5779	-	-
1500	-	6381	-	-	-	-	-	6379	-	-
1600	-	6445	-	-	-	-	-	6439	-	-

The first important observation refers to the results of point-mass model. The mathematical assumptions made during the development of its theory eliminate the lateral deviation of projectile. The trajectory of point-mass model is ordinarily a plane curve. In other words, when there is no transverse wind, this model does not take in account any transverse motion of the projectile, as experimentally observed. On the other hand, modified point-mass and 6-DOF trajectories are never plane, they show a sideways deviation of the projectile from the vertical plane of release. Even when there is no transverse wind, the projectile describes a three-dimensional trajectory. This is precisely why these two models are so much better than point-mass, they allow us to measure and predict transverse motion of the projectile.

Analyzing Tables 1 and 2, one can note that, for releases made with elevation up to 1200ml (67º30'), relative errors in trajectories of the point-mass model are inferior to 3% for ranges and 7% for maximum heights. Although at first this seems to be a good precision level, it corresponds to absolute errors of 212m in range and 187m in maximum height, which may be a poor achievement. Considering a tolerance of 30m in range and 10m in maximum height as reasonable values for artillery firings, only releases with elevation angles less than 400ml (22º30') would be considered reproducing trajectories accurately in this model.

However, in the trajectories simulations with the modified point-mass model, rotation speed becomes an important parameter. In this model, there is transverse motion. For elevation angles up to 500ml (28º7'30"), which are releases of tenser trajectories, the results obtained with this model are closer to experimental data than those obtained by the point-mass model. Applying the same criteria to modified point-mass trajectories (a tolerance of 30m in range and 10m in maximum height), results are satisfactory for initial elevation angles up to 500ml (28º7'30"), now with transverse motion.

In what concerns 6-DOF trajectories, there is a precision compromise between range and maximum height, that is, when ranges are very accurate, the results for maximum heights are less accurate, and vice-versa. However, from a generic point of view, maximum relative errors are notably smaller than those obtained by the previous models, as far as results could be checked. This is a very significant parameter in the analysis of the model's efficiency as a whole. Discarding punctual deviations from experimental data to give more importance to general performance, 6-DOF model gives the best predictions in the entire universe of possible trajectories with elevation angles up to 1300ml (73º). Besides, its theoretical development is the broadest among the three models in study, so it can be useful in the understanding of projectile dynamics in a greater degree than point-mass and modified point-mass. It also predicts transverse motion with greater accuracy than modified point-mass.

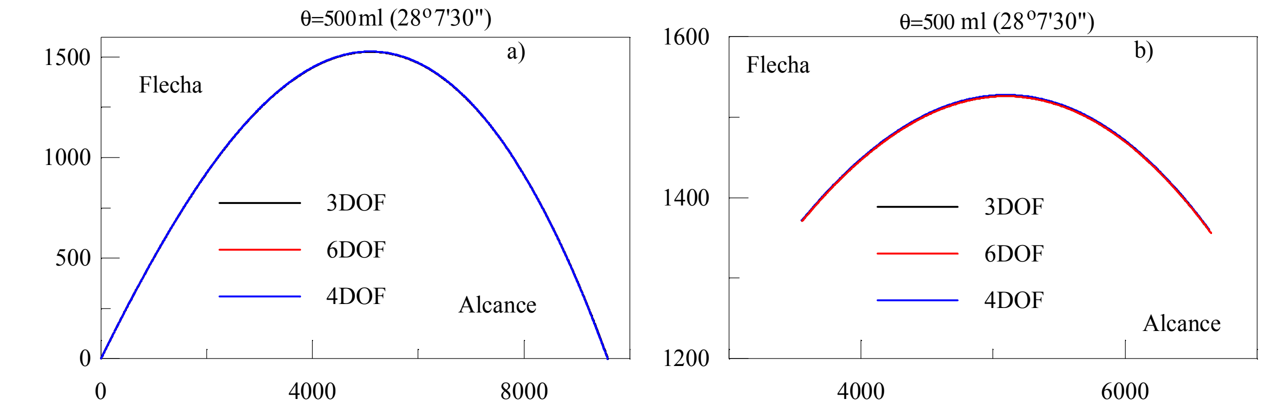

Figure 2 - Trajectories predicted by the models for θ = 500ml (28º7'30").

Fig. 2 and 3 exhibit some trajectories predicted by the models. The trajectories in Fig. 2 correspond to a small release angle, while the trajectory in Fig. 3 refers to a high release angle. Analysis of Fig. 2 shows that the trajectories almost coincide for θ = 28 7'30". There are practically no differences in range and height obtained from the different models with this release angle.

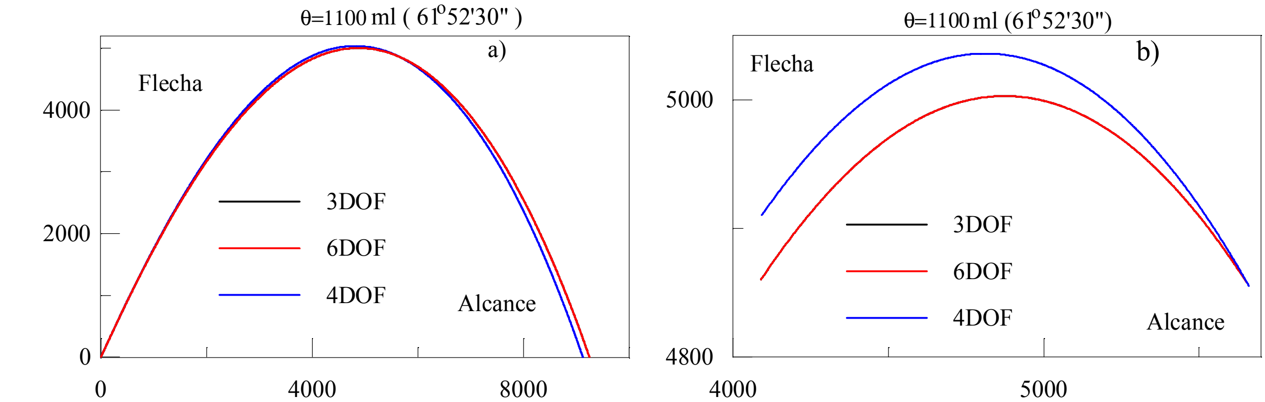

Figure 3: Trajectories predict by the models for θ=1100ml (61º52'30")

In Fig. 3, one verifies that the trajectory predicted by the modified point mass model reveals effective differences in range and height regarding other models. The point-mass model and the model of six degrees of freedom have coincident trajectories and predict smaller heights. Moreover, the position of the point of maximum height was predicted by these two models at slightly lower ranges in comparison to the prediction of modified point-mass model.

6. Conclusion

In this work were simulated, with three models of distinct complexity, ballistic trajectories of projectiles stabilized by rotation. The simulations have shown that the time of computation is not an important limitation to utilizing any of the mathematical models. Releases with high elevation angles, resulting in trajectories with great altitudes, have obtained a processing time of about ten seconds, whilst releases with small elevation angles and smaller altitudes resulted in computation times of about of five seconds.

The models obtained general results very close to data foreseen by shot tables. For tense trajectories, with small angles of release, the modified point-mass model gave the best results. However, in a global evaluation, one can affirm that the six-degrees-of-freedom model is the best representation of the physical phenomenon of ballistic flight, since largest adjustment of its results to experimental data in every interval of angles of release was achieved.

Finally, it has been verified that none of the models, together with the adopted ballistic coefficients, have reproduced well the experimental data for release angles greater than 70º. Some hypotheses, for instance, the perfect alignment of the projectile with the tube in the moment of release, and the complete set of aerodynamic coefficients need to be evaluated thoroughly, in order to correct the predictions for larger release angles.

7. References

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