# ANALISYS OF THE ANALYTICAL RESOLUTION OF THE RADIATIVE TRANSFER EQUATION BY FOR AN OBLIQUELY INCIDENT COLLIMATED BEAM ONTO THE SURFACE

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***Abstract.*** *A methodology to solve the Radiative Transfer Equation (RTE) is presented. It is assumed a scattering semitransparent material (STM). A one-dimensional geometry without azimuthally symmetry function of inclination of the incident beam is used to transform a problem without azimuthal symmetry to a set of problems with azimuthal symmetry, where the radiation intensity field is described as Fourier series expansion and the phase function is described as Legendre's polynomial.*

***Keywords:*** *Radiative Transfer Equation, Differential – Integral Equations, Semitransparent Material, Discrete Ordinates Method.*

## 1. Introduction

The semitransparent materials that can absorbing, emitting and anisotropically scattering the thermal radiation, has a very large industrial application. In this paper an analytical formulation to solve the Radiative Transfer Equation, RTE, with or without azimuthally symmetry condition are presented. Chandrasekar (1960) and Ozisik (1973) had used this formulation to transform a problem without azimuthally symmetry to a set of problems with azimuthal symmetry, using Dirac's function, where the radiation intensity field is described as Fourier series expansion and the phase function is described as Legendre's polynomial. Results are present to isotropic and anisotropic cases.

## 2. Mathematical formulation

In this analysis a collimated beam has an oblique incidence onto a semitransparent material, that is, is necessary to solve the Radiative Transfer Equation without azimuthal symmetry and an additional difficult is increase to solve this equation. The formulation must to consider the spatial asymmetry and the integration around the azimuth angle is not possible. The adopted formulation transform the without azimuthally symmetry problem in a set of azimuthally symmetry problems. Two formulations are presented for isotropic and anisotropic scattering.

### 2.1. Isotropic Scattering

An isotropic medium scattered uniform for all directions, phase function, $p$=1. Others assumptions are:

- One-dimensional geometry.
- No interface.
- A surface is submitted to a collimated incident radiation beam.

The Radiative Transfer Equation is given by:

$$\mu \frac{\partial I(\tau,\mu,\theta)}{\partial \tau} + I(\tau,\mu,\theta) = g(\tau) + \frac{\omega}{4\pi} \int_{\phi'=0}^{2\pi} \int_{\mu'=-1}^{+1} I(\tau,\mu',\theta')\, d\mu'\, d\phi' \tag{1}$$

where g(τ) is the no homogeneous term and the boundary conditions are:

$$I(0,\mu,\phi) = F_1(\mu,\phi) \ to \ \mu > 0$$
$$I(\tau_0,\mu,\phi) = F_2(\mu,\phi) \ to \ \mu < 0 \quad (2)$$

Using an auxiliary problem to Eq. (1), where the integral term is eliminated, and writing the radiative intensity in two parts, $I_0^+$ *e* $I_0^-$ , with the boundary conditions given by Eq. (2), the result is:

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = \frac{\omega}{4\pi} G_0(\tau) + \frac{\omega}{2} \int_{\mu'=-1}^{+1} I(\tau,\mu')\, d\mu' \quad (3)$$

With the boundary conditions:

$$I(0,\mu) = 0 \ to \ \mu > 0$$
$$I(\tau_0,\mu) = 0 \ to \ \mu < 0 \quad (4)$$

### 2.2. Anisotropic Scattering

Anisotropic scattering is when there is a nonuniform distribution of scattered directions. Generally, scattering behavior provide complicated solutions. Other assumption hypotheses can be described as:

- One-dimensional geometry
- No generation energy term
- Obliquely incident beam in $\tau = 0$
- No irradiation in $\tau = \tau_0$
- The phase function is represented like Legendre polynomials.

The RTE, for this problem, is given by:

$$\mu \frac{\partial I(\tau,\mu,\theta)}{\partial \tau} + I(\tau,\mu,\theta) = \frac{\omega}{4\pi} \int_{\phi'=0}^{2\pi} \int_{\mu'=-1}^{+1} p(\mu_0)\ I(\tau,\mu',\theta')\, d\mu'\, d\phi' \quad (5)$$

where $p(\mu_0)$ is the phase function and written like a Legendre polynomial:

$$p(\mu_0) = \sum_{n=0}^{N} a_n P_n(\mu_0)\ ,\ a_0 = 1 \quad (6)$$

$\mu_0$ is the angle between $\Omega$ and $\Omega$'. The boundary conditions are:

$$I(0,\mu,\phi) = F_1(\mu,\phi) \ to \ \mu > 0$$
$$I(\tau_0,\mu,\phi) = 0 \quad to \ \mu < 0 \quad (7)$$

Using an homogeneous auxiliary problem, the RTE is given by:

$$\mu \frac{\partial I_1(\tau,\mu,\phi)}{\partial \tau} + I_1(\tau,\mu,\phi) = \frac{\omega}{4\pi} \left[ \int_0^{2\pi} \int_{-1}^{+1} \left( p(\mu_0)\ F_1(\mu,\phi) \exp\left(-\tau/\mu\right) \right) d\mu'\ d\phi' \right] +$$
$$+ \frac{\omega}{4\pi} \int_0^{2\pi} \int_{\mu'=-1}^{+1} p(\mu_0)\ I_1(\tau,\mu',\phi')\, d\mu'\, d\phi' \quad (8)$$

with the boundary conditions given by:

$$I\left(0,\mu,\phi\right)= 0 \ to \ \mu > 0$$
$$I\left(\tau_0,\mu,\phi\right)= 0 \ to \ \mu < 0 \tag{9}$$

Equation (8) is separated in a set of equations with azimuthally symmetry, the phase function is represented by Legendre polynomials and the radiative intensity is written as series Fourier expansion.

**3. Fourier expansion**

A function $F_1$ is written in terms of the external incidence intensity, $f_1$, incident angle $\theta_1$ and azimuthally angle $\phi_1$:

$$F_1\left(\mu,\phi\right)= f_1\,\delta\left(\mu_1-\mu\right)\,\delta\left(\phi_1-\phi\right) \tag{10}$$

The sine term in Fourier series represent a diffuse boundary. Thus the radiative intensity can be written only in terms the cosine azimuthally angle, function of the coordinate position and the incidence angle. Let Eq. (8) and substituting the radiative intensity and the boundary conditions, we has (Özisik, 1973):

$$\mu\,\frac{\partial\left[\sum_{k=0}^{\infty} I^k\left(\tau,\mu\right)\cos k\left(\phi_1-\phi\right)\right]}{\partial\tau}+\sum_{k=0}^{\infty} I^k\left(\tau,\mu\right)\cos k\left(\phi_1-\phi\right)=$$
$$=\frac{\omega}{4\pi}\int_{\phi'=0}^{2\pi}\int_{\mu'=-1}^{+1} F_1\left(\mu',\phi'\right)\exp\left(-\tau/\mu\right)\left[\sum_{m=0}^{N}\sum_{n=m}^{N}\left(2-\delta_{0m}\right)a_n^m\,P_n^m\left(\mu\right)P_n^m\left(\mu'\right)\cos m\left(\phi-\phi'\right)\right]d\mu'\,d\phi'+ \tag{11}$$
$$+\frac{\omega}{4\pi}\int_{\phi'=0}^{2\pi}\int_{\mu'=-1}^{+1} d\mu'\,d\phi\left[\sum_{m=0}^{N}\sum_{n=m}^{N}\left(2-\delta_{0m}\right)a_n^m\,P_n^m\left(\mu\right)P_n^m\left(\mu'\right)\cos m\left(\phi-\phi'\right)\right]\left[\sum_{k=0}^{\infty} I^k\left(\tau,\mu\right)\cos k\left(\phi_1-\phi\right)\right]$$

Equation (11) can be transformed in a set of equations where the cosine term in the summation is linear independent:

$$\mu\,\frac{\partial\left[I^k\left(\tau,\mu\right)\right]}{\partial\tau}+I^k\left(\tau,\mu\right)=0 \ to \ k > N$$
$$\mu\,\frac{\partial\left[I^k\left(\tau,\mu\right)\right]}{\partial\tau}+I^k\left(\tau,\mu\right)=\frac{\omega}{4\pi}\,f_1\,\exp\left(-\tau/\mu_1\right)\left[\sum_{n=k}^{N}\left(2-\delta_{0k}\right)a_n^k\,P_n^k\left(\mu\right)P_n^k\left(\mu_1\right)\right]+ \tag{12}$$
$$+\frac{\omega}{2}\sum_{n=k}^{N} a_n^k\,P_n^k\left(\mu\right)\int_{\mu'=-1}^{+1} P_n^k\left(\mu'\right)I^k\left(\tau,\mu'\right)d\mu' \ to \ k\le N$$

The boundary conditions are:

$$I\left(0,\mu\right)= 0 \ to \ \mu > 0$$
$$I\left(\tau_0,\mu\right)= 0 \ to \ \mu < 0 \tag{13}$$

The values for $k$ and $n$ are given as: $k$ = 0, 1, 2, ..., $N$ and $n = k$, $k$+1, $k$+2, ..., $N$.

The solution of Eq. (12) is difficult function of the integral term. The discrete ordinates method is used to substitute the integral term by a summation and the radiative transfer equation can be written as:

$$\mu\,\frac{\partial\left[I^k\left(\tau,\mu\right)\right]}{\partial\tau}+I^k\left(\tau,\mu\right)=\frac{\omega}{4\pi}\,f_1\,\exp\left(-\tau/\mu_1\right)\left[\sum_{n=k}^{N}\left(2-\delta_{0k}\right)a_n^k\,P_n^k\left(\mu\right)P_n^k\left(\mu_1\right)\right]+$$
$$+\frac{\omega}{2}\sum_{n=k}^{N} a_n^k\,P_n^k\left(\mu\right)\left[\sum_{j=1}^{N} C_j\,\left(P_n^k\left(\mu_j\right)+I^k\left(\tau,\mu_j\right)+P_n^k\left(-\mu_j\right)+I^k\left(\tau,-\mu_j\right)\right)\right] \ to \ k\le N \tag{14}$$

The discrete direction $\mu_I$ is used in Eq.(14) and dividing this equation by $\mu_I$, the RTE is described as:

$$\frac{\partial\left[I^k(\tau,\mu_I)\right]}{\partial\tau}+\frac{I^k(\tau,\mu_I)}{\mu_I}=\frac{\omega}{4\pi}\,f_1\,\frac{\exp\left(-\tau/\mu_1\right)}{\mu_I}\left[\sum_{n=k}^{N}(2-\delta_{0k})\,a_n^k\,P_n^k(\mu_i)P_n^k(\mu_1)\right]+$$
$$+\frac{\omega}{2\mu_I}\sum_{k=0}^{N}a_n^k\,P_n^k(\mu_I)\left[\sum_{j=1}^{N}C_j\,\left(P_n^k(\mu_j)+I^k(\tau,\mu_j)+P_n^k(-\mu_j)+I^k(\tau,-\mu_j)\right)\right] \quad (15)$$

The constant, $f_1$, is writing as a collimated intensity, $I_0\,d\omega_0$, and rewriting Eq. (12) in transmittance terms, it can be show that:

$$\frac{\partial\left[T(\tau)\right]}{\partial\tau}=T(\tau)\left[\frac{-1}{\mu_I}+\frac{\omega}{2\mu_I}\sum_{k=0}^{N}a_n^k\,P_n^k(\mu_I)\left[\sum_{j=1}^{N}C_j\,\left(P_n^k(\mu_j)+P_n^k(-\mu_j)\right)\right]\right]$$
$$+\frac{\omega}{4\pi}\,\frac{\exp\left(-\tau/\mu_1\right)}{\mu_I}\left[\sum_{k=0}^{N}(2-\delta_{0k})\,a_n^k\,P_n^k(\mu_I)P_n^k(\mu_1)\right] \quad (16)$$

## 4. Analysis

In this section two anisotropy cases are analyzed, $p$=1. A normal incidence onto the sample/medium are considered. A modified Henyey–Greenstein phase function is used (Nicolau, 1994):

$$p(\mu_i,\mu_j)=f_2\left[f_1\,p_{HG,g_1}(\mu_i,\mu_j)+(1-f_1)\,p_{HG,g_2}(\mu_i,\mu_j)\right]+(1-f_2) \quad (17)$$

In Table 1 are shown the parameters (albedo, optical thickness and extinction coefficient) taken in account in this analysis.

Table 1. Parameters values in anisotropic case 1.

| albedo | Extinction coefficient [1/ m] | Optical Length | Directions | Inclination | Thickness [ m ] | $f_1$ | $g_1$ | $f_2$ | $g_2$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.916 | 2.965 | 11.86 | 24 | 0° | 0.004 | 0 | 0 | 1 | 0.9 |

Figure 1 present the errors to different volumes to the solution by discrete ordinate method and volume control taken the analytical solution to eigenvalues and eigenvectors (Buiar and Moura 2003). When the volumes number increase the numerical results close to analytical solution. Figure 2 shows transmittances and reflectances calculated for this condition.

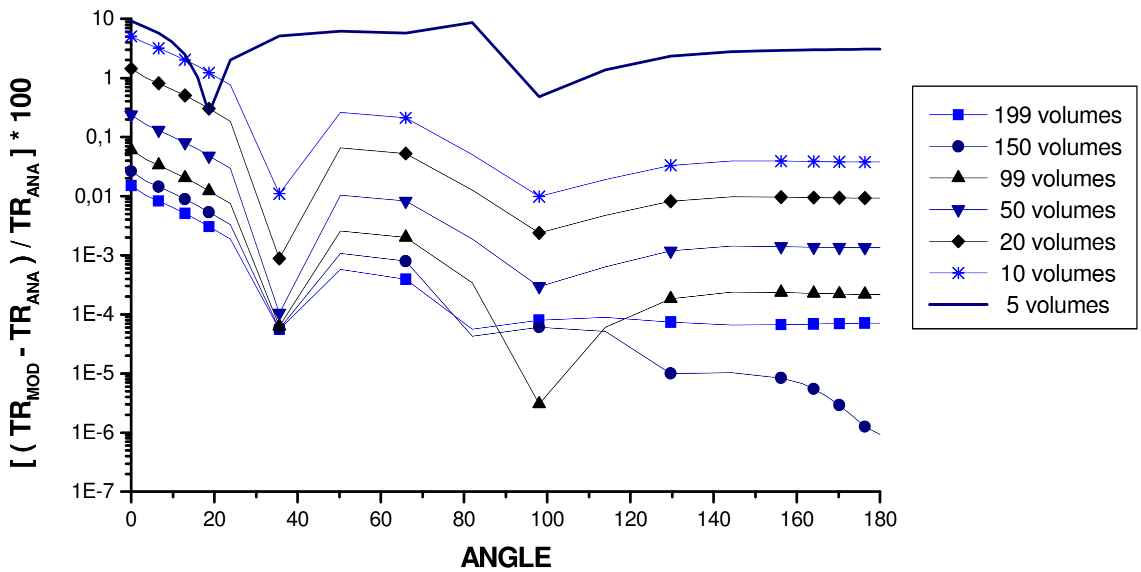


Figure 1. Errors value to anisotropic case 1.

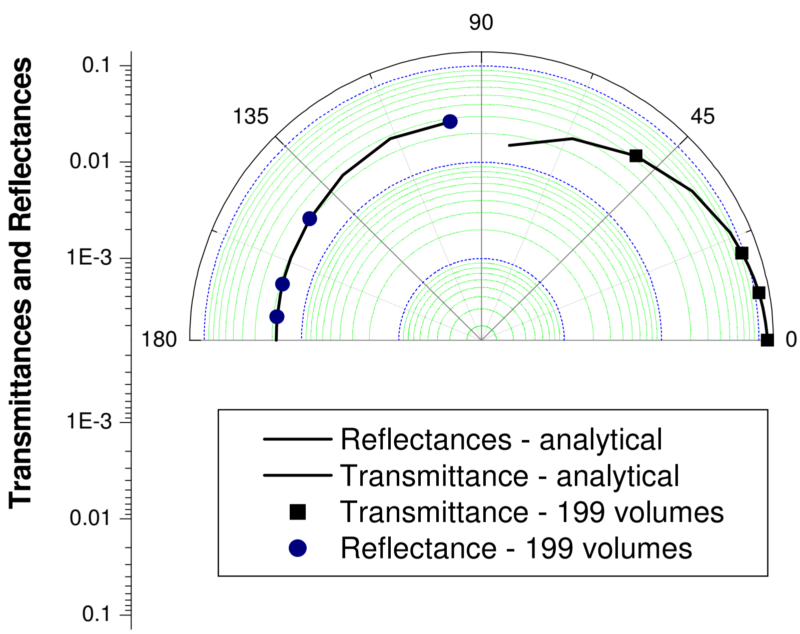


Figure 2 – Transmittance and Reflectance to anisotropic 1 case.

Table 2 presents the parameters used to the second anisotropic case, and the Figs. 3 and 4 show the results.

Table 2. Parameters values to the in case 2.

| albedo | Extinction coefficient [1/m] | Optical Thickness | Directions | Inclination [ ° ] | Thickness [m] | $f_1$ | $g_1$ | $f_2$ | $g_2$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.916 | 2.965 | 11.86 | 24 | 0 | 0.004 | 0.975 | 0.918 | 0.9 | -0.9 |

Figures 3 and 4 present the errors in relation to an analytical solution and the conclusion is the same that the last case. Figure 5 presents the computational time for a PC-P4 with 2.8 GHz.

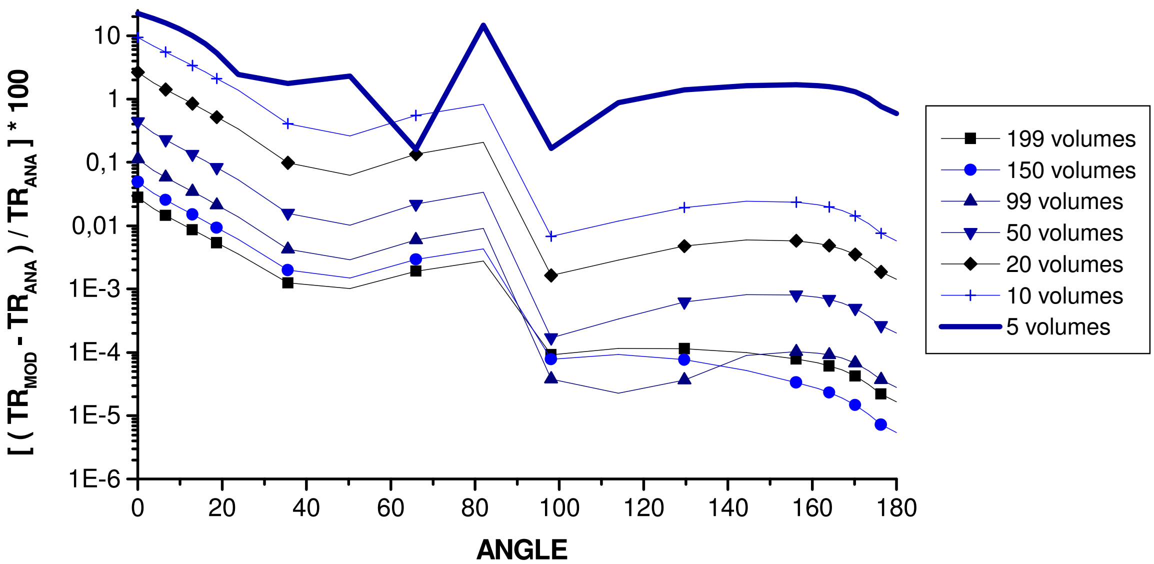


Figure 3. Errors to anisotropic case 2.

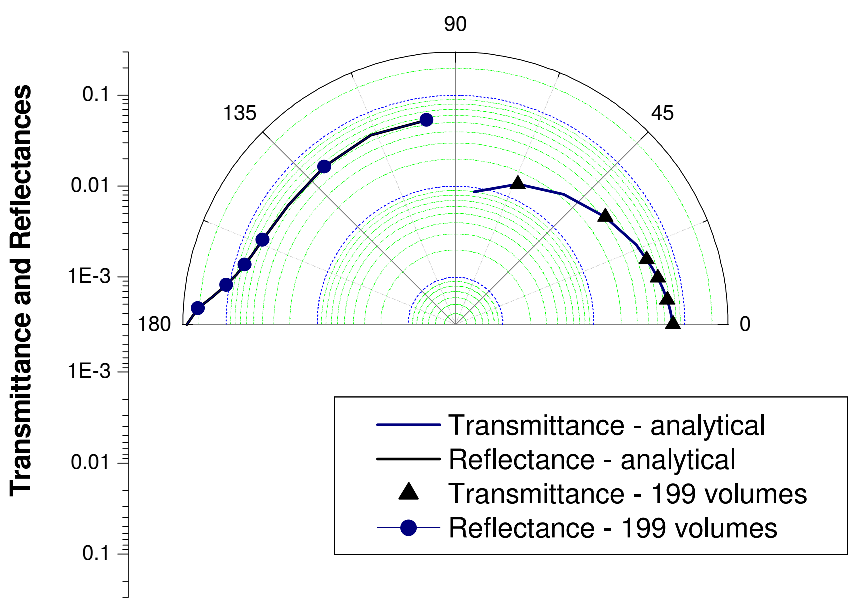


Figure 4. Reflectance and Transmittance values to anisotropic case 2.

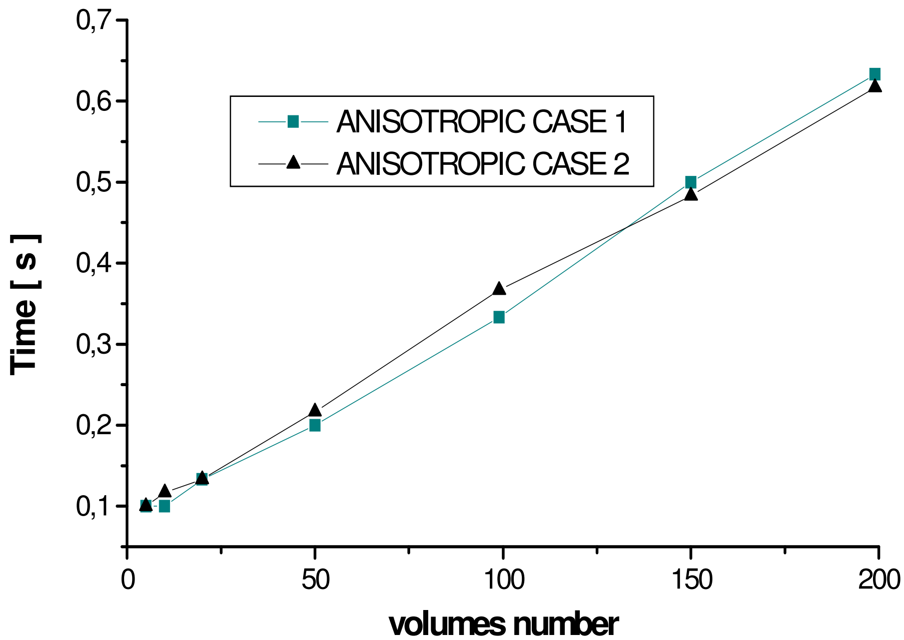


Figure 5. Comparison between computational time to two anisotropic cases for a PC-P4, 2.8 GHz.

## 5. Conclusion

The methodology presents in this paper are analyzed to solve the radiative transfer equation in semitransparent materials (MST), in a one-dimensional geometry, without azimuthally symmetry condition. The case tests allow us to determine the precision of the Ordinate Discrete Method and evaluate the number of volumes and the time necessary for the analyses.

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