EXACT ANALYTICAL SOLUTION OF STRATIFIED THREE-PHASE LAMINAR FLOW IN A HORIZONTAL RECTANGULAR DUCT

Jian Su
Nuclear Engineering Program, COPPE, Universidade Federal do Rio de Janeiro, CP 68509, Rio de Janeiro, 21945, Brazil
sujian@con.ufrj.br

Allen T. Chwang
Department of Mechanical Engineering,The University of Hong Kong, Pokfulam Road, Hong Kong, China
atchwang@hkucc.hku.hk

Abstract. *Exact analytical solution of stratified three phase laminar flow in a horizontal duct of rectangular cross-section is obtained by applying the classical finite integral transform. The governing partial differential equations are transformed into an infinite set of ordinary differential equations by applying integral transform in the transverse direction. Each set of three coupled second-order linear ordinary differential equations are solved analytically, with the solution coefficients being determined by solving a set of 6 linear algebraic equations resulting from the boundary and interface conditions in the vertical direction. Quantities of interests such as wall and interfacial friction factors are consequently determined. The obtained solution of velocity distribution in three layers can be used for further stability and regime transition analysis.*

Keywords: *Three-phase flow, stratified flow, laminar flow, analytical solution, integral transform*

1. Introduction

It is surprising that very limited research work has been dedicated to the study of three-phase gas-liquid-liquid system, despite its obvious practical importance. Açikgõgz et al (1992) reported experimental results for a wide range of flow rates and observed 10 different flow patterns for the case of horizontal oil-water-air flow. Lee et al (1993) observed and classified seven flow patterns, which are similar to the case of two-phase flows. Açikgõgz et al (1992) and Lahey et al (1992) derived drift-flux expressions for horizontal three phase flows in order to predict the phase volumetric fractions for air-water-oil flows. Three phase flows in horizontal or slightly inclined pipes were also studied by Nuland et al (1991) and Stapelberg et al (1990, 1991a & b), where as the vertical case was considered by Chen et al (1990) and Pieshko & Sharma (1990). The work of Stapelberg treated primarily slug flow where the flow is sufficiently chaotic that the liquid phase is essentially a mixture of the two liquid used, water and oil.

Of particular relevance to the present work are the works by Taitel et al (1995) and Khor et al (1997). The first one presented a theoretical approach to solve the three phase stratified flow, where the purpose was to calculate the levels of the bottom liquid layer and the second layer. The solution method adopted by them was to reduce the phasic momentum balance equations to two simultaneous equations by eliminating the (equal) pressure gradients. Then the combined equations are converted to dimensionless form by introducing simplified expressions for the shear stresses; the equations are then solved iteratively.

Khor et al. (1997) applied the three-phase model to estimate phase holdups in three phase stratified flow. They estimates the phase holdups by comparing the pressure drops in each phase which are derived from the phasic momentum balances equations. The desired solution is the point where all three phases have the same pressure gradient. Using selected correlations for shear stresses, the water and oil levels were systematically adjusted until the pressure gradients for the three phases were equal, the final values of the liquid levels giving the phase holdups as required.

The solution of single phase laminar flow of an incompressible fluid in a horizontal rectangular duct can be found in classical textbooks on fluid mechanics. The solution of laminar two-phase flow in a horizontal rectangular duct was obtained by Tang and Himmelblau (1963). In the present work, we present the exact analytical solution of three-phase stratified co-current laminar flow in a horizontal duct of rectangular cross section by using finite Fourier transform.

2. The Mathematical Model

We consider here the hydrodynamics of steady, fully developed, laminar, co-current, stratified, three-phase (usually gas-liquid-liquid) flow in a horizontal duct of rectangular cross section, as illustration in Figure 1. The fluids are considered to be Newtonian with constant viscosities and densities. The effect of surface tension is not considered. Due to the symmetry of the flow with respect to the plane $x = 0$, only a half of duct, $0 \le x \le a$, is considered. The governing equations are written as

$$\frac{\partial^2 u_m(x,y)}{\partial x^2} + \frac{\partial^2 u_m(x,y)}{\partial y^2} = \frac{1}{\mu_m}\frac{dp}{dz}, \quad \text{in} \quad 0 < x < a, \quad h_{m-1} < y < h_m, \quad m = 1,2,3 \tag{1}$$

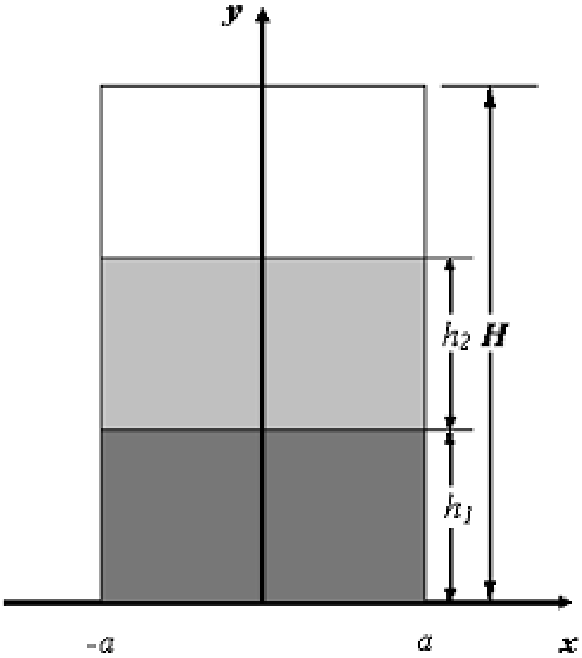

Figure 1. Three-phase stratified flow in a duct of rectangular cross-section.

where $u_m(x,y)$ is the velocity of m-th fluid, μ_m its viscosity, dp/dz is the longitudinal pressure gradient, x is the transversal coordinate, y the vertical coordinates, z the longitudinal coordinate, a is the half-width of the duct, $h_0(=0)$ is the bottom of duct, and $h_3(=H)$ is the height of the rectangular duct, h_1 and h_2 are positions of the fluid interfaces.

The governing equations (1) are to be solved subject to the following boundary and interface conditions

$$\frac{\partial u_m(x,y)}{\partial x} = 0, \quad \text{in} \quad x=0, \quad h_{m-1} < y < h_m, \quad m=1,2,3 \tag{2}$$

$$u_m(x,y) = 0, \quad \text{in} \quad x=a, \quad h_{m-1} < y < h_m, \quad m=1,2,3 \tag{3}$$

$$u_1(x,y) = 0 \quad \text{in} \quad y=0, \quad 0<x<a \tag{4}$$

$$u_m(x,y) = u_{m+1}(x,y) \quad \text{in} \quad 0<x<a, \quad y=h_m, \quad m=1,2 \tag{5}$$

$$\mu_m \frac{\partial u_m(x,y)}{\partial y} = \mu_{m+1} \frac{\partial u_{m+1}(x,y)}{\partial y} \quad \text{in} \quad 0<x<a, \quad y=h_m, \quad m=1,2 \tag{6}$$

$$u_3(x,y) = 0 \quad \text{in} \quad y=H, \quad 0<x<a. \tag{7}$$

3. Integral Transform Solution

The velocity distribution is represented by the following orthogonal expansion:

$$u_{m,i}(x,y) = \sum_{i=1}^{\infty} \overline{u}_{m,i}(y)\, \psi_i(x), \quad m=1,2,3 \tag{8}$$

where the transformed velocity potentials are defined by

$$\overline{u}_{m,i}(y) = \int_0^a \psi_i(x)\, u_m(x,y)\, dx, \quad m=1,2,3 \tag{9}$$

The normalized eigenfunctions $\psi_i(x)$ are defined by the following auxiliary problem

$$\frac{d^2\psi_i(x)}{dx^2} + \lambda_i^2\, \psi_i(x) = 0 \tag{10}$$

$$\frac{d\psi_i(x)}{dx} = 0 \quad \text{at} \quad x=0 \quad \text{and} \quad \psi_i(x) = 0 \quad \text{at} \quad x=a \tag{11}$$

The eigenfunctions and eigenvalues corresponding to the problem defined by Eqs.(10 to 11) can be readily obtained as:

$$\psi_i(x) = \sqrt{\frac{2}{a}}\, \cos(\lambda_i x), \quad \lambda_i = \frac{(2i-1)\pi}{a}, \quad i = 1, 2, ... \tag{12}$$

Taking the finite Fourier transform of the governing equations (1), the boundary conditions (4 to 7), and the interfacial conditions (5) and (6), we have the following ordinary differential equations for the transform potentials with the respective boundary and interfacial conditions:

$$\frac{d^2\overline{u}_{m,i}(y)}{dy^2} - \lambda_i^2\, \overline{u}_{m,i}(y) = \overline{g}_{m,i}; \quad m = 1, 2, 3 \tag{13}$$

$$\overline{u}_{1,i}(y) = 0; \qquad \text{at} \qquad y = 0 \tag{14}$$

$$\overline{u}_{m,i}(y) = \overline{u}_{m+1,i}(y); \qquad \text{at}\, y = h_m, \quad m = 1, 2 \tag{15}$$

$$\mu_m \frac{d\overline{u}_{m,i}(y)}{dy} = \mu_{m+1} \frac{d\overline{u}_{m+1,i}(y)}{dy}; \qquad \text{at} y = h_m, \quad m = 1, 2 \tag{16}$$

$$\overline{u}_{3,i}(y) = 0; \qquad\qquad \text{at} y = H \tag{17}$$

where

$$\overline{g}_{m,i} = \frac{1}{\mu_m}\frac{dp}{dz}\int_0^a \psi_i(x)dx = \sqrt{\frac{2}{a}}\frac{2(-1)^{i+1}}{(2i-1)\pi}\frac{1}{\mu_m}\frac{dp}{dz} \tag{18}$$

The general solutions of the second-order linear ordinary differential equations (13) are as follows

$$\overline{u}_{1,i}(y) = A_{1,i}\cosh(\lambda_i y) + B_{1,i}\sinh(\lambda_i y) - \frac{\overline{g}_{1,i}}{\lambda_i^2}, \tag{19}$$

$$\overline{u}_{2,i}(y) = A_{2,i}\cosh(\lambda_i y) + B_{2,i}\sinh(\lambda_i (h_1 - y)) - \frac{\overline{g}_{2,i}}{\lambda_i^2}, \tag{20}$$

$$\overline{u}_{3,i}(y) = A_{3,i}\cosh(\lambda_i y) + B_{3,i}\sinh(\lambda_i (H - y)) - \frac{\overline{g}_{m,i}}{\lambda_i^2}. \tag{21}$$

The constants $A_{1,i}, B_{1,i}, A_{2,i}, B_{2,i}, A_{3,i}, B_{3,i}$ are to be determined by applying the boundary and interface conditions. For clarity, the six linear algebraic equations for i-th transformed velocities are written in detail:
at $y = 0$:

$$A_{1,i} = \frac{\overline{g}_{1i}}{\lambda_i^2} \tag{22}$$

at $y = h_1$:

$$A_{1,i}\,\mu_1 \sinh(\lambda_i h_1) + B_{1,i}\,\mu_1 \cosh(\lambda_i h_1) - A_{2,i}\,\mu_2 \sinh(\lambda_i h_1) + B_{2,i}\,\mu_2 = 0 \tag{23}$$

$$A_{1,i}\, \cosh(\lambda_i h_1) + B_{1,i}\, \sinh(\lambda_i h_1) - A_{2,i}\, \cosh(\lambda_i h_1) = \left(\frac{\overline{g}_{1i} - \overline{g}_{2i}}{\lambda_i^2}\right) \tag{24}$$

at $y = h_2$:

$$A_{2,i}\,\mu_2 \sinh(\lambda_i h_2) + B_{2,i}\,\mu_2 \cosh(\lambda_i h_2) - A_{3,i}\,\mu_3 \sinh(\lambda_i h_2) - B_{3,i}\,\mu_3 \cosh(\lambda_i h_2) = 0 \tag{25}$$

$$A_{2,i}\, \sinh(\lambda_i h_2) - B_{2,i}\, \cosh(\lambda_i (h_1 - h_2)) + \; + A_{3,i}\, \sinh(\lambda_i h_2) - B_{3,i}\, \cosh(\lambda_i (H - h_2)) = \left(\frac{\overline{g}_{2i} - \overline{g}_{3i}}{\lambda_i^2}\right) \tag{26}$$

at $y = h_3$:

$$B_{3,i}\, \cosh(\lambda_i H) = \frac{\overline{g}_{3i}}{\lambda_i^2} \tag{27}$$

The choice of the particular form of the general solution results in the direct obtention of two constants, $A_{1,i}$ and $A_{3,i}$. The system of the four remaining linear algebraic equations is to be solved to obtain the constants $B_{1,i}, A_{2,i}, B_{2,i}, B_{3,i}$ for each Fourier component of the velocity. The solution of the linear system is straightforward.

Therefore, the exact solution of the problem is given by:

$$u_1(x,y) = \sum_{i=1}^{\infty} \left[A_{1,i} \cosh(\lambda_i y) + B_{1,i} \sinh(\lambda_i y) - \frac{\overline{g}_{1i}}{\lambda_i^2} \right] \cdot \left[\sqrt{\frac{2}{a}} \cos(\lambda_i x) \right] \tag{28}$$

$$u_2(x,y) = \sum_{i=1}^{\infty} \left[A_{2,i} \cosh(\lambda_i y) + B_{2,i} \sinh(\lambda_i (h_1 - y)) - \frac{\overline{g}_{2i}}{\lambda_i^2} \right] \cdot \left[\sqrt{\frac{2}{a}} \cos(\lambda_i x) \right] \tag{29}$$

$$u_3(x,y) = \sum_{i=1}^{\infty} \left[A_{3,i} \cosh(\lambda_i y) + B_{3,i} \sinh(\lambda_i (H - y)) - \frac{\overline{g}_{3i}}{\lambda_i^2} \right] \cdot \left[\sqrt{\frac{2}{a}} \cos(\lambda_i x) \right] \tag{30}$$

4. Quantities of Interests

We now proceed to obtain quantities of interests of the laminar three-phase stratified flow for each region: the phasic volumetric flow rate, the wall shear stresses, the interfacial shear stresses, and the friction factors.
Volumetric flow rates:

$$Q_m = \int_{h_{m-1}}^{h_m} \int_0^a u_m(x,y) dx dy, \quad m = 1,2,3 \tag{31}$$

The average velocities are given by

$$U_m = \frac{Q_m}{a(h_m - h_{m-1})} \quad m = 1,2,3 \tag{32}$$

Wall shear stress:

$$\tau_{1w} = \frac{\mu_1}{a + h_1} \left(\int_0^a \frac{\partial u_1(x,y)}{\partial y} dx - \int_0^{h_1} \frac{\partial u_1(x,y)}{\partial x} dy \right) \tag{33}$$

$$\tau_{2w} = \frac{\mu_2}{h_2 - h_1} \left(- \int_{h_1}^{h_2} \frac{\partial u_2(x,y)}{\partial x} dy \right) \tag{34}$$

$$\tau_{3w} = \frac{\mu_3}{a + H - h_1} \left(- \int_0^a \frac{\partial u_1(x,y)}{\partial y} dx - \int_{h_1}^{H} \frac{\partial u_3(x,y)}{\partial x} dy \right) \tag{35}$$

Interfacial shear stress:

$$\tau_{12} = \frac{\mu_2}{a} \left(\int_0^a \frac{\partial u_2(x,y)}{\partial y} dx \right), \quad \tau_{23} = \frac{\mu_3}{a} \left(\int_0^a \frac{\partial u_3(x,y)}{\partial y} dx \right) \tag{36}$$

Wall friction factor:

$$f_1 = \frac{2\tau_{1w}}{\rho_1 U_1^2}, \quad f_2 = \frac{2\tau_{2w}}{\rho_2 U_2^2}, \quad f_3 = \frac{2\tau_{3w}}{\rho_3 U_3^2} \tag{37}$$

Interfacial friction factor:

$$f_{12} = \frac{2\tau_{12}}{\rho_2 (U_2 - U_1)^2}, \quad f_{23} = \frac{2\tau_{23}}{\rho_3 (U_3 - U_2)^2} \tag{38}$$

5. Results and Discussion

For a given set of pressure gradient, dp/dz, and the liquid levels h_1 and h_2, the velocity distributions in the three regions can be determined, as well as the quantities of interests. However, in practical applications, the pressure gradient and the liquid levels are unknowns, while the volumetric flow rates of each phase are given. In this case, a set of 3 transcendental equations must be solved to determine the pressure gradient and liquid levels

$$Q_m\left(\frac{dp}{dz}, h_1, h_2\right) - Q_{m,given} = 0 \quad m = 1,2,3$$

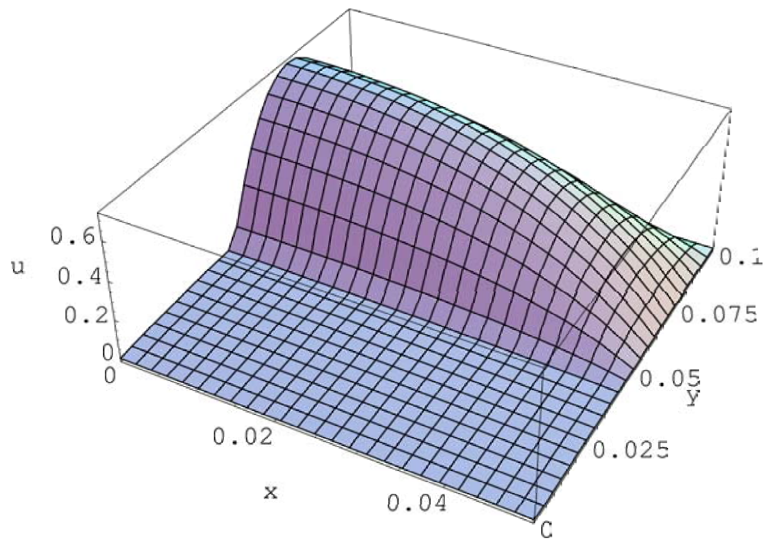

Figure 2. Velocity distribution of three phase flow in a duct of rectangular cross-section.

As an example, we consider a water-oil-air three phase horizontal flow, being fluids 1, 2, 3 respectively. The properties of water and air are taken as of the standard conditions: ρ_1=1000 kg/m^3, μ_1 = 0.001 kg/m·s, ρ_3=1.21 kg/m^3, μ_2 = 1.81×10^{-5} kg/m·s. The properties of oil are taken as ρ_2=800 kg/m^3, μ_2 = 0.002 kg/m·s. The half width of the duct is 0.05 m and the height is 0.1 m.

For $Q_1 = 2.0 \times 10^{-5}$ m^3/s (water), $Q_2 = 1.5 \times 10^{-5}$ m^3/s (oil), and $Q_3 = 0.001$ m^3/s, the pressure gradient is calculated to be -0.373 Pa, the water level h_1 being 0.0316 m, and the oil (total liquid) level h_2 being 0.0442. For this case, the superficial gas velocity U_{gs} = 0.2 n/s, the total liquid superficial velocity U_{ls} = 0.007 m/s, the Reynolds numbers of three phases are $Re_1 = 980$, $Re_2 = 1906$, and $Re_3 = 1716$. The composite velocity distribution of the three phase flow in the whole cross-section of the horizontal duct is shown in Fig.2. The water and oil velocity distributions are detailed in Figs. 3 and 4 respectively.

The liquid holdup (h_2/H) as a function of the total liquid superficial velocity is shown in Fig.5, for a constant gas superficial velocity $U_{gs} = 0.2m/s$. The liquid holdup as a function of the gas superficial velocity, for a given constant total liquid superficial velocity $U_{ls} = 0.007m/s$, is shown in Fig.6. Fig.7 shows the wall friction factor of fluid 1, f_{1w}, as a function of the Reynolds number Re_1; Fig.8 shows the wall friction factor of fluid 2 f_{2w} and the liquid-liquid interfacial friction factor f_{12} as a function of the Reynolds number Re_2; and Fig.9 shows the wall friction factor of fluid 3 f_{3w} and the gas-liquid interfacial friction factor f_{23} as a function of the Reynolds number Re_3. The Fanning friction factor of laminar flow in a circular pipe, $16/Re$, is plotted in the three figures for comparison. It can be from Fig.8 that the circular pipe friction factor $16/Re$ is a good approximation for the liquid-liquid interfacial friction factor for $Re_2 > 1000$. Fig.9 shows that the gas phase wall friction factor can be reasonably approximated by the circular pipe solution $16/Re$. However, the liquid-liquid interfacial friction factor is significantly different from the fluid 2 wall friction factor, and the gas-liquid interfacial friction factor is also significantly different from the gas wall friction factor.

6. Conclusion

An exact analytical solution is obtained for stratified three phase laminar flow in a horizontal duct of rectangular section by using integral transform technique. Parameters of interest such as interfacial friction factors and wall friction factors are obtained also in analytical form. The analytical solution obtained can be used to provide friction factors needed in one-dimensional modelling of stratified three phase flow, such as that described in Taitel et al. (1995) and in Khor et al.(1997).

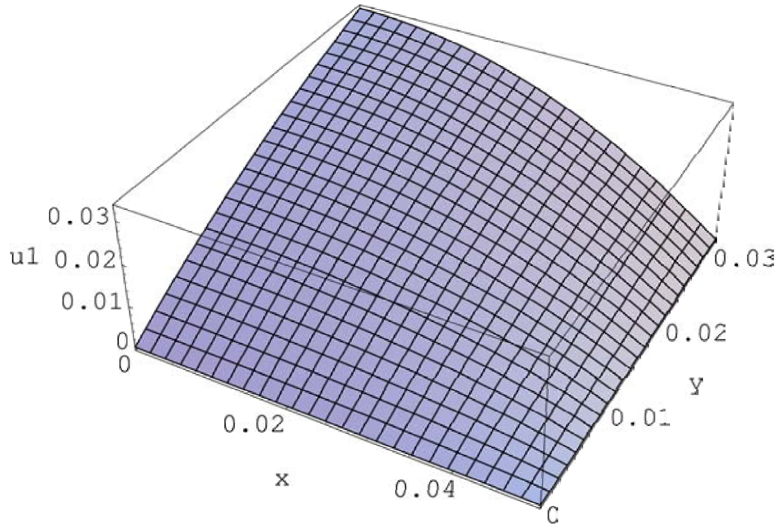

Figure 3. Velocity distribution of water.

7. Acknowledgements

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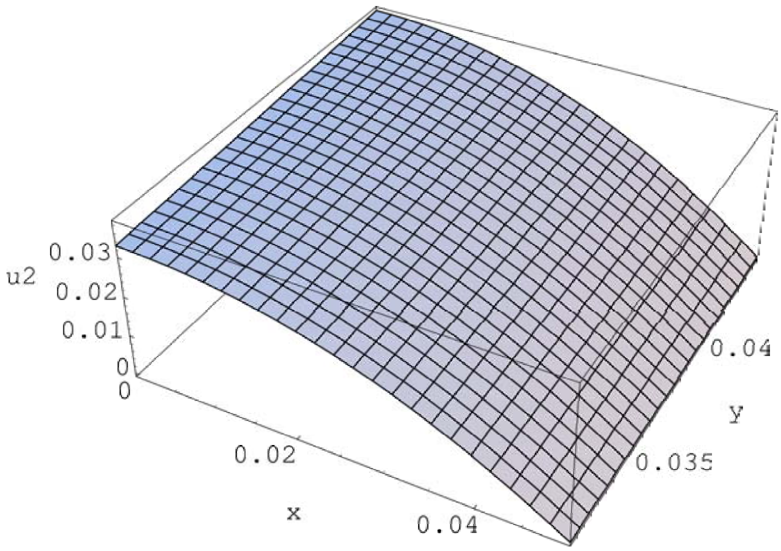

Figure 4. Velocity distribution of oil.

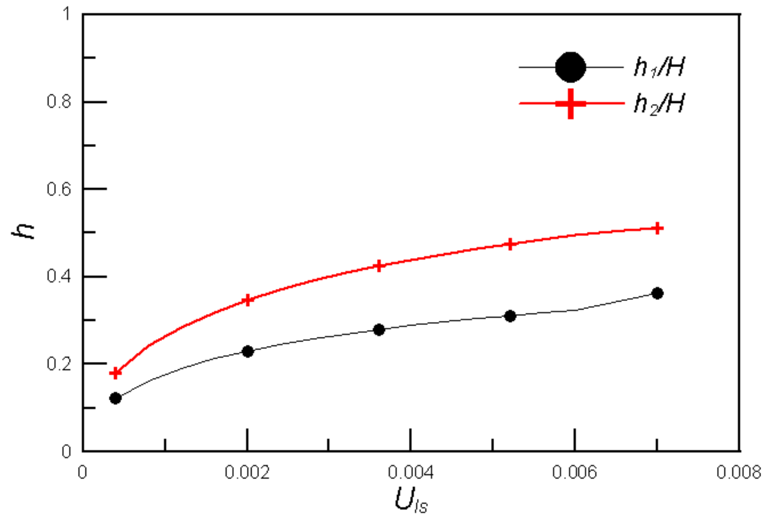

Figure 5. The liquid holdup as a function of the total liquid superficial velocity

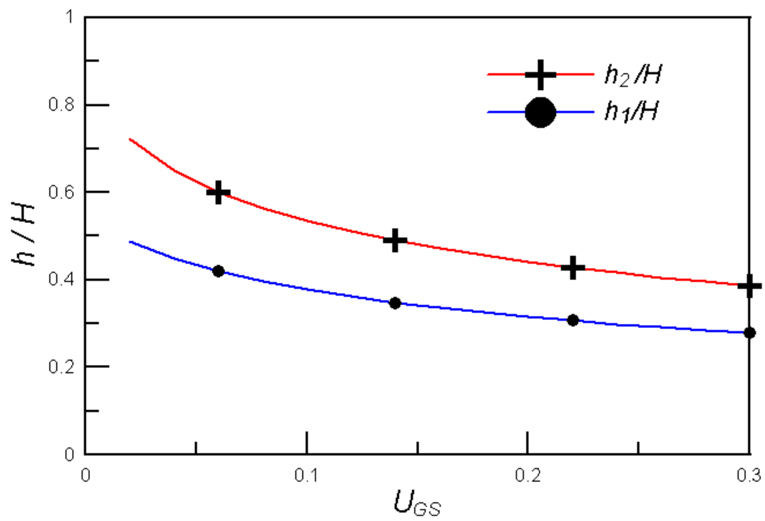

Figure 6. The liquid holdup as a function of the gas superficial velocity

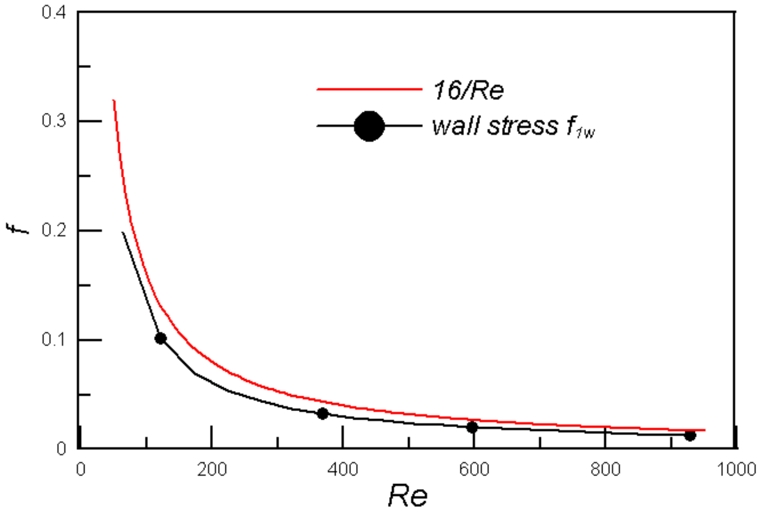

Figure 7. The wall friction factor of fluid 1 as a function of Reynolds number

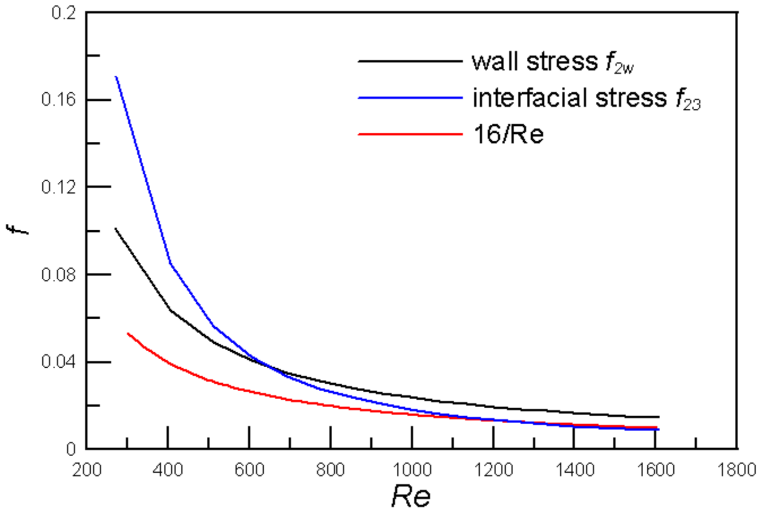

Figure 8. The wall friction factor of fluid 2 and liquid-liquid interfacial friction factor as a function of Reynolds number

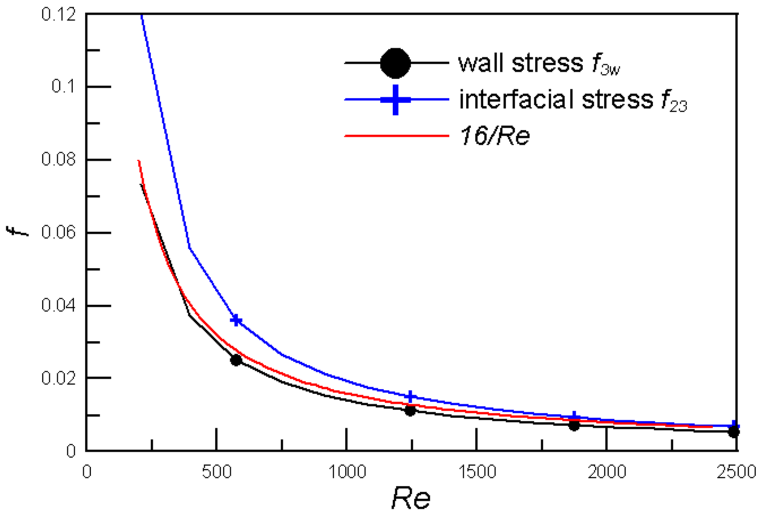

Figure 9. The gas wall friction factor and gas-liquid interfacial friction factor as a function of Reynolds number