Numerical Stability Analysis of Composite Plates Thermally Stiffened by Finite Element Method

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Abstract. *In the study of laminated plates it is interesting to consider the thermal stiffening effects caused by the difference between cure and operation temperatures. This paper presents a finite element formulation , according to the Reissner-Mindlin theory, of the buckling problem of laminated composite plates under thermal residual stresses. A geometric stiffness matrix is obtained and applied to several critical load buckling problems of composite plates. The numerical results are compared to results available in the literature.*

Keywords*: finite elements, buckling, laminated, thermal analysis, plate stability*

1. Introduction

In the prediction of plate structures behavior the critical buckling load is of vital importance. For anisotropic laminates residual stresses caused by difference of cure and service temperatures may have an effect in the critical buckling load.

The thermal residual stresses appear because each lamina has different expansion coefficients in fiber direction and normal to fiber direction. When a plate undergoes a difference of temperature there are two possible behaviors, it can deform freely or be under thermal residual stresses if it is restrained. In symmetric plates thermal residual stress appears only when the plate is constrained or reinforced.

The subject of composite plates under residual thermal stress has potential for applications in aeronautical structures. Contrary to the first feeling, laminates can be tailored such that there is an increase of the buckling load for some range of service temperature.

This study is performed with a quadratic triangular finite element based on Reissner-Mindlin theory, proposed by Sze *et al.* (1997) and which has its formulation described in detail by Goto (2002), where also its capability was enhanced to deal with membrane behavior.

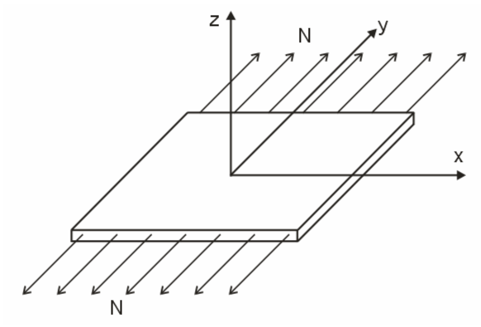


Figure 1. Global coordinate system considered

2. Formulation and Results

This work is subdivided in three parts: the buckling problem description, finite element geometric stiffness formulation and numerical results.

2.1. Buckling Problem Description and Formulation

The theory of buckling deals principally with conditions under which equilibrium ceases to be stable. A structure is said to be buckled when it loses stability. This can be seen when two or more possible stability positions appear for a certain load, and probably none of them are stable.

Buckling of laminated composite plates can happen caused basically by three types of loads: compression in fiber direction, compression normal to fiber direction (in plate plane) and by shear (also in plate plane). Any of this loads and any combination of them will have a critical load, which is the load when the plate loses stability and become buckled.

To calculate the critical load we will use the total potential energy criterion (El Naschie, 1990), which is especially interesting for finite element analysis. This criterion states that the second derivate of total potential energy of the plate with respect to each displacement is zero when the applied load approaches the critical load.

$$\pi = \frac{1}{2}\int_V \{\varepsilon\}^T \{\sigma\} dV + \frac{1}{2}\int_V \{\gamma\}^T \{\tau\} dV - \int_A \{u\}^T \{f\} dA \tag{1}$$

In Eq.(1), $\{\varepsilon\}$ is the strain vector, $\{\sigma\}$ is the stress vector, $\{\gamma\}$ is the out of plane shear strain vector, $\{\tau\}$ is the out of plane shear stress vector, $\{u\}$ is the displacement vector, $\{f\}$ is the external forces vector and π is the total potential energy. V denotes the volume and A the area of the plate.

So, to evaluate the critical load we apply a load case of interest. Appling a multiplier on the applied stresses, with the second derivate of π we can calculate the multiplier factor of the loads, which is the critical load.

2.2. Finite element formulation for buckling analysis

Sze *et al.* (1997) proposed the AST6, a six node triangular element (Figure 2) based on Reissner-Mindlin theory for bending of plates. This element uses quadratic interpolation for bending degrees of freedom (DOF) and linear interpolation for out of plane shear strain. Lucena Neto *et al.* (2001) formulated the element explicitly, i.e., using analytical integration and added membrane DOF to it.

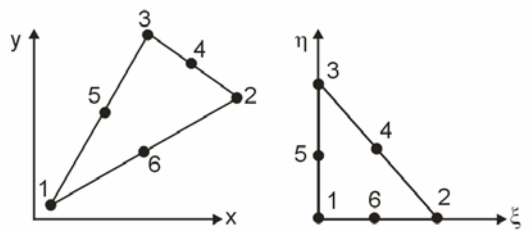


Figure 2. AST6 in global coordinates and local coordinates

To solve the buckling problem we will consider the pre-buckling stress state. Immediately before the buckling of the plate acting stresses are due to a loading identical to the critical load. In order to determine this stress state two linear independent static stress analyses must be done: one for thermal loading and another for mechanical loading. Knowing the stress state we can compute the geometric stiffness matrices associated to each loading and then finally find the solution of the resulting eigenvalue problem.

The linear static finite element analysis development described in Goto's (2002) master thesis is omitted here. The following quantities defined in the local plate coordinate system (Figure 1) are needed:

$$\left(A_{ij}, B_{ij}, D_{ij}, E_{ij}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{ij}\left(1, z, z^2, z^3\right) dz \,,\; i, j = 1,2,6 \tag{2}$$

$$\left(G_{ij}, F_{ij}\right) = k_{ij} \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{ij}\left(1, z\right) dz \,,\; i, j = 4,5 \tag{3}$$

The matrices A, B, D and E are the stiffness matrices for the in plane behavior, G and F are the out of plane shear stiffness matrices, Q is the layer constitutive matrix, N is the number of layers, z_{k+1} is the height at top of layer k and z_k is the height at bottom of layer. With these quantities, we can define the generalized loads:

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{L\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \\ [D] & [E] \end{bmatrix} \begin{Bmatrix} \{\varepsilon_m\} \\ \{\kappa\} \end{Bmatrix} \tag{4}$$

$$\begin{Bmatrix} \{Q\} \\ \{T\} \end{Bmatrix} = \begin{bmatrix} [G] \\ [F] \end{bmatrix} \{\gamma\} \tag{5}$$

In Eq.(4) and Eq.(5), $\{N\}$, $\{M\}$, $\{L\}$, $\{Q\}$ and $\{T\}$ are the stress resultants vectors, $\{\varepsilon_m\}$ is the strain vector at the mid-plane of the plate (z=0) and $\{\kappa\}$ is the flexural strain vector.

For the buckling problem, we must use the nonlinear parts of the strain vectors. So, defining the strains vectors with $\{\varepsilon^L\}$ and $\{\gamma^L\}$ as the linear parts, and $\{\varepsilon^N\}$ and $\{\gamma^N\}$ as the nonlinear parts (Verri, 2000):

$$\begin{aligned} \{\varepsilon\} &= \{\varepsilon^L\} + \{\varepsilon^N\} \\ \{\gamma\} &= \{\gamma^L\} + \{\gamma^N\} \end{aligned} \tag{6}$$

$$\{\varepsilon^L\} = \{\varepsilon_m\} + z\{\kappa\} = \begin{Bmatrix} u,_x \\ v,_y \\ u,_y + v,_x \end{Bmatrix} + z \begin{Bmatrix} \theta_{y'x} \\ -\theta_{x'y} \\ \theta_{y'y} - \theta_{x'x} \end{Bmatrix} \tag{7}$$

$$\{\varepsilon^N\} = \begin{Bmatrix} \tfrac{1}{2}\left(u,_x^2 + v,_x^2 + w,_x^2 + 2z\left(u,_x\theta_{y'x} - v,_x\theta_{x'x}\right) + z^2\left(\theta_{y'x}^2 + \theta_{x'x}^2\right)\right) \\ \tfrac{1}{2}\left(u,_y^2 + v,_y^2 + w,_y^2 + 2z\left(u,_y\theta_{y'y} - v,_y\theta_{x'y}\right) + z^2\left(\theta_{y'y}^2 + \theta_{x'y}^2\right)\right) \\ u,_x u,_y + v,_x v,_y + w,_x v,_y + z^2\left(\theta_{y'y}\theta_{y'x} + \theta_{x'y}\theta_{x'x}\right) + z\left(u,_x\theta_{y'y} + u,_y\theta_{y'x} - v,_x\theta_{x'y} - v,_y\theta_{x'x}\right) \end{Bmatrix} \tag{8}$$

$$\{\gamma^L\} = \begin{Bmatrix} w,_x + \theta_y \\ w,_y - \theta_x \end{Bmatrix} \tag{9}$$

$$\{\gamma^N\} = \begin{Bmatrix} u,_x\theta_y - v,_x\theta_x + z\left(\theta_{y'x}\theta_y + \theta_{x'x}\theta_x\right) \\ u,_y\theta_y - v,_y\theta_x + z\left(\theta_{y'y}\theta_y + \theta_{x'y}\theta_x\right) \end{Bmatrix} \tag{10}$$

With:

$$i,_j = \frac{\partial i}{\partial j} \tag{11}$$

More details on these definitions and plate theory can be found in Reddy (1945).

2.2.1 Linear initial pre-buckling thermal stress analysis

The thermal load stress resultants vector are given by

$$\left(\{N_T\},\{M_T\}\right)=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}[Q]\{\alpha\}\Delta T(1,z)\,dz \qquad (12)$$

The thermal load vectors are defined by Eq.(13), where the subscript m and b refer to membrane and bending displacement degrees of freedom. $[B_m]$ and $[B_b]$ are matrices relating the strain vectors with the nodal DOF vectors by Eq.(14), according to Goto (2002), $\{\alpha\}$ is the thermal expansion coefficients vector and ΔT is the difference between work and cure temperature.

$$\{f_{Tm}\}=\iint_A [B_m]^T\,dxdy\{N_T\},\{f_{Tb}\}=\iint_A [B_b]^T\,dxdy\{M_T\} \qquad (13)$$

$$\{\varepsilon_m\}=[B_m]\{d_m\},\ \{\kappa\}=[B_b]\{d_b\} \qquad (14)$$

Assembling the system equations of equilibrium, we solve the linear system:

$$[K]\{d\}=\{f_T\} \qquad (15)$$

Matrix $[K]$ is the linear stiffness of the plate, defined by Eq.(16) from Eq.(1), while $\{d\}$ contains the structure nodal displacements.

$$\{d\}^T[K]\{d\}=\int_V\{\varepsilon\}^T\{\sigma\}dV+\int_V\{\gamma\}^T\{\tau\}dV \qquad (16)$$

With the displacement vector we can calculate the deformation at any point inside an element by Eq.(7) and Eq.(14), and from the deformation with Eq.(4) and Eq.(5) we get the stress resultants and the difference between these resultants (no subscript) and the thermal resultants (with subscript T), are the following thermal residual stress resultants (subscript R).

$$\begin{Bmatrix}\{N_R\}\\ \{M_R\}\end{Bmatrix}=\begin{Bmatrix}\{N\}\\ \{M\}\end{Bmatrix}-\begin{Bmatrix}\{N_T\}\\ \{M_T\}\end{Bmatrix},\begin{Bmatrix}\{L_R\}\\ \{Q_R\}\\ \{T_R\}\end{Bmatrix}=\begin{Bmatrix}\{L\}\\ \{Q\}\\ \{T\}\end{Bmatrix} \qquad (17)$$

2.2.2 Linear initial pre-buckling applied loading analysis

The pre-buckling stress analysis for the applied loading vector, $\{f_P\}$, is carried out with the solution of the linear system given by Eq.(15). This analysis is distinct from the thermal initial stress analysis because we need separate stress states. Again, with the displacement vector $\{d\}$, from Eq.(7), Eq.(14) and Eq.(4) we calculate the stress resultants for the pre-buckling state, denoted by the subscript P, which are $\{N_P\},\{M_P\},\{L_P\},\{Q_P\}$ and $\{T_P\}$.

2.2.3 Buckling eigenvalue problem

With all the pre-buckling information available we can now formulate the linear buckling equation. The total potential energy for buckling is given by

$$\pi=\frac{1}{2}\int_V\{\varepsilon\}^T\left(\{\sigma_P\}+\{\sigma_T\}\right)dV+\frac{1}{2}\int_V\{\gamma\}^T\left(\{\tau_P\}+\{\tau_T\}\right)dV \qquad (18)$$

This equation has only initial pre-buckling stresses. From Eq.(16) into Eq.(18), and with (19) we have:

$$\{d\}[K_G]\{d\} = \int_V \{\varepsilon^N\}^T \{\sigma\} dV + \int_V \{\gamma^N\}^T \{\tau\} dV \tag{19}$$

$$\pi = \frac{1}{2}\left(\{d\}^T [K]\{d\} + \{d\}^T [K_{GT}]\{d\} + \{d\}^T [K_{GP}]\{d\}\right) \tag{20}$$

Where $[K_G]$ is the geometric stiffness matrix, which is composed of two parts, associated to thermal and mechanical loading, according to the formulation presented by Almeida and Hansen (1997), and also used by Verri (2000).

A geometric stiffness matrix is said to be inconsistent if the interpolation functions used to generate it are different from the ones used for the linear stiffness. For AST6, to avoid mesh locking effect (Reddy, 1993), the shear deformations use linear interpolation functions for the linear stiffness (Sze *et al.*, 1997), which results from a quadratic interpolation where the mid side nodes displacements are average of edge displacements. Here we use quadratic interpolation functions for all the terms in the geometric stiffness matrix. This is done because the mesh locking effect does not occur with the non-linear deformation terms that appear in the geometric stiffness matrix.

Defining the vector $\{e_g\}$, for the AST6:

$$\{e_g\}^T = \{u_{,x} \quad u_{,y} \quad v_{,x} \quad v_{,y} \quad w_{,x} \quad w_{,y} \quad \theta_{y',x} \quad \theta_{y',y} \quad -\theta_{x',x} \quad -\theta_{x',y} \quad \theta_y \quad -\theta_x\} \tag{21}$$

$$\{e_g\}^T = [[N_1] \quad [N_2] \quad [N_3] \quad [N_4] \quad [N_5] \quad [N_6]]\{d\} = [NN]\{d\} \tag{22}$$

$$[N_i]^T = \begin{bmatrix} n_{i',x} & n_{i',y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_{i',x} & n_{i',y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_{i',x} & n_{i',y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -n_{i',x} & -n_{i',y} & 0 & -n_i \\ 0 & 0 & 0 & 0 & 0 & 0 & n_{i',x} & n_{i',y} & 0 & 0 & n_i & 0 \end{bmatrix} \tag{23}$$

In Eq.(23) n_i are the element displacement interpolation functions. Recalling the definition of the stress resultants, (Eq.(2) to Eq.(4)), we define $[\Psi]$:

$$[\Psi] = \begin{bmatrix} N_{xx} & N_{xy} & 0 & 0 & 0 & 0 & M_{xx} & M_{xy} & 0 & 0 & Q_x & 0 \\ N_{xy} & N_{yy} & 0 & 0 & 0 & 0 & M_{xy} & M_{yy} & 0 & 0 & Q_x & 0 \\ 0 & 0 & N_{xx} & N_{xy} & 0 & 0 & 0 & 0 & M_{xx} & M_{xy} & 0 & Q_y \\ 0 & 0 & N_{xy} & N_{yy} & 0 & 0 & 0 & 0 & M_{xy} & M_{yy} & 0 & Q_y \\ 0 & 0 & 0 & 0 & N_{xx} & N_{xy} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{xy} & N_{yy} & 0 & 0 & 0 & 0 & 0 & 0 \\ M_{xx} & M_{xy} & 0 & 0 & 0 & 0 & L_{xx} & L_{xy} & 0 & 0 & T_x & 0 \\ M_{xy} & M_{yy} & 0 & 0 & 0 & 0 & L_{xy} & L_{yy} & 0 & 0 & T_y & 0 \\ 0 & 0 & M_{xx} & M_{xy} & 0 & 0 & 0 & 0 & L_{xx} & L_{xy} & 0 & T_x \\ 0 & 0 & M_{xy} & M_{yy} & 0 & 0 & 0 & 0 & L_{xy} & L_{yy} & 0 & T_y \\ Q_x & Q_x & 0 & 0 & 0 & 0 & T_x & T_y & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_y & Q_y & 0 & 0 & 0 & 0 & T_x & T_y & 0 & 0 \end{bmatrix} \tag{24}$$

Finally we obtain

$$[K_G] = \iint_A \left([NN]^T [\Psi][NN] \right) dxdy \tag{25}$$

Eq.(25) can be formulated for any initial stress state which is explicit in the stress resultants appearing in [Ψ], Eq.(24). In our case $[K_{GT}]$ is due to thermal stress resultants and $[K_{GP}]$ is due to stress resultants from mechanical loading. A numerical integration is used for Eq.(25), similarly to what was done for the linear stiffness by Sze *et al.* (1997).

The variation of π, Eq.(20), leads to

$$\frac{\partial^2 \pi}{\partial \{d\}^2} = \det\left([K] + \lambda_T [K_{GT}] + \lambda_P [K_{GP}] \right) = 0 \tag{26}$$

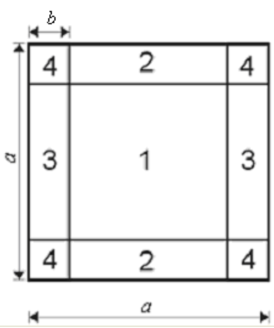
In the eigenproblem of Eq.(26), two buckling load factors appear, λ_P, associated with the mechanical applied loading and λ_T, associated with the thermal loading. Three buckling problem can be defined depending on which of these factors are free and which at fixed values. When only λ_T is free (λ_P fixed) we have the thermal buckling problem with initial mechanical stress. When only λ_P is free (λ_T fixed) we have the mechanical buckling problem with thermal initial stress. When none of them is fixed, λ_P nor λ_T, they should be merged in only one eigenvalue and the geometric stiffness matrices should be added. There are as much critical loads and buckling modes as nodal displacement degrees of freedom, but we need the first (minimum module) critical load, corresponding to the minimum module eigenvalue. Finally solving the eigenproblem for the first eigenvalue we find the critical load by multiplying the applied load by the corresponding eigenvalue.

2.3 Analysis and results

Two study cases are presented, the first case being for the purpose of finite element geometric stiffness validation. The second case considers a reinforced composite plate under thermal residual stresses and uniform compression load. The material properties are defined in Table (1), and the plate lamination sequences are illustrated in Figure (3) for case two.

Table 1. Material properties and dimensions considered

Property	Value for case 1	Value for case 2
Longitudinal modulus of elasticity, E_1	123.4 GPa	154.5 GPa
Transversal modulus of elasticity, E_2	2 GPa	11.13 GPa
In-plane Poisson's ratio, υ_{12}	0.29	0.304
Transversal shear modulus, G_{23}	2.9 GPa	3.36 GPa
Transversal shear modulus, G_{13}	5.8 GPa	6.98 GPa
In-plane shear modulus, G_{12}	5.8 GPa	6.98 GPa
Longitudinal thermal expansion coefficient, α_1	-	-0.17 X 10^{-6}/ºC
Transversal thermal expansion coefficient, α_2	-	23.1 X 10^{-6}/ºC
Lamina side, *a*	1000 mm	360 mm
Base plate lamina thickness, t_b	1 mm	0.15 mm
Reinforced plate lamina thickness, t_r	-	0.15 mm



Laminate	Base Plate	Reinforced
1	$[\theta_4]$	$[\theta_4]$
2	$[\theta_4]$	$[90/90/\theta_4/90/90$
3	$[\theta_4]$	$[0/0/\theta_4/0/0]$
4	$[\theta_4]$	$[90/0/\theta_4/0/90]$

Figure 3. Plate geometry and lamination sequences

2.3.1 Unidirectional compression buckling case

A uniform compression is applied in the X-direction (Figure 1), without thermal loading, therefore λ_T is zero. Once Eq.(26) is solved for the buckling load factor, λ_P, its product by the applied distributed load gives the buckling load. The plate is simply supported. The results in Table 2, obtained with a 15x15 finite element mesh (900 elements), show a very good agreement when compared to converged finite element solutions obtained with MSC-Nastran, with 100x100 (10000 elements) finite element mesh (Isoni, 2004).

Table 2. Buckling load factor

Laminate	AST6	Nastran (Isoni)	Error
[0]	94.11	94.07	0.05%
[90]	35.22	35.19	0.08%
[45]	19.81	20.43	3.00%
[0/90/0]	53.15	53.17	0.04%
[90/0/90]	28.41	28.42	0.02%
[0/45/0]	75.69	75.72	0.03%
[90/45/90]	35.79	35.69	0.28%
[45/0/45]	23.95	24.60	2.64%
[45/90/45]	23.64	24.26	2.58%
[45/-45/45]	31.63	32.24	1.88%
[0/90]s	34.25	34.28	0.09%
[90/0]s	27.47	27.49	0.06%
[0/45]s	58.24	58.32	0.14%
[90/45]s	33.69	33.75	0.18%
[45/0]s	29.69	30.22	1.77%
[45/90]s	29.18	29.69	1.72%
[45/-45]s	46.65	47.13	1.03%

2.3.2 Temperature and compression buckling case

This case considers the buckling of reinforced plate due to an uniform distributed load applied to its horizontal edges (Figure 3), which corresponds to the eigenproblem solution for λ_P from Eq.(26), with λ_T fixed and equal to unity. The plate is simply supported on all four edges, and the in plane displacements normal to each edge are constrained, except at the superior edge where the load is applied. To calculate the thermal residual stresses the plate is considered unconstrained, but to prevent rigid body motion some displacement degrees of freedom must be properly constrained.

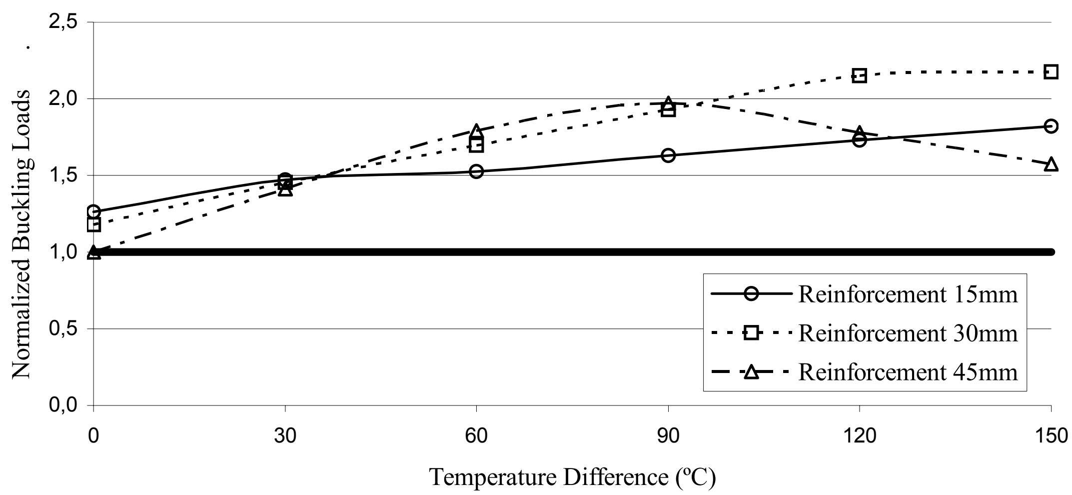


Figure 4. Normalized critical load by temperature difference

Three different reinforcement widths, b, are used, 15, 30 and 45mm, and θ is 90º. The results shown in Figure (4) are the buckling load factors normalized with respect to buckling load factor of the uniform thickness plate (without reinforcement). The thickness t_b is scaled for each value of b to maintain the plate volume the same as the uniform thickness plate. Therefore, values greater than one indicate buckling load factors greater than that of the uniform plate. The temperature difference in X-axis of Figure (4) is in fact negative ($-\Delta T$).

3. Acknowledgements

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