BOUNDARY ELEMENT FUNDAMENTAL TENSORS FOR AXISYMMETRIC LINEAR ELASTICITY PROBLEMS

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Abstract. *This paper presents the Boundary Element formulation for axysimmetric elastic problems. The Kelvin solution, which uses a unitary concentrated load in an infinite elastic domain to generate the fundamental solution, is taken into account. Initially, the three-dimensional problem expressed in cartesian coordinates is transformed to cylindrical ones. In a second step, the mathematical expressions are integrated in the "θ" variable, changing it in a two-dimensional model. In this mathematical strategy, elliptic integrals and its derivatives occur, which are manipulated to achieve the fundamental stresses. Hard solving singular integrals would need to be solved using traditional collocation of source points on the boundary. Here the positions of source points are external to physical domain, avoiding singularities.*

Keywords: *Axysimmetric, Elliptic Integral, Boundary Element Method.*

1. Introduction

It is very common and very important the solution of axisymmetric problems in engineering. Frequently, these problems are associated with energy storage and transfer. Simple examples are vessels, pipes, rotors, etc. The physical variable associated can be displacement, temperature, eletric potential among many others. These cases allow very interesting simplification in their mathematical treatment. Only the revolution section is enough to represent all the body, producing economy and simplicity in any engineering analysis. However, it is necessary to be aware that many problems have axisymmetrical form, but the boundary conditions are not axisymmetric. These problems are not really axisymmetric. Special mathematical models need to be applied in these cases, which depend also on the angular coordinates.

The application of Boundary Element Method in axisymmetric problems is very advantageous and the numerical results are very accurate, since the integral equations were previously solved to the angular coordinate. In general, this procedure in field scalar problems has been employed with success, due to the ease the ellyptic equations acquirement. However, in elasticity problems, the field is vectorial, producing cumbersome eliptic integrals. The derivative of these eliptic integrals, more specifically to acchieve the fundamental stress tensor, has restricted the more wide application of the Boundary Element procedure. Papers presented by Kermanidis (1975), Cruse et al (1977), Mayr & Neureiter (1977) and Brebbia et al (1984) show expressions to fundamental strain tensors, but the fundamental stress tensor was not explicited. Beyond this fact, (Kermanidis, 1975) was the unique author to present the three-dimensional fundamental solution expressed in cilindrical coordinate system.

In this paper, all fundamental tensors that compose the boundary element formulation, usually named kernels, are explicited. It is necessary to point out the deduction of the fundamental stress tensor expression because, as mentioned before, not even Brebbia (1981) and Kithe (1995) in specific books about axisymmetric problems presented it. There is another interesting article, written by Mayr et al (1980), which axissymetric problems with arbitrary boundary conditions are analysed. Many impression errors in the text were detected, rendering the tensor expression presented an unsafe reference for implementation.

Another difficulty usually found in axisymmetric problems is the singularity of boundary integrals, resulting of the coincidence among source point and field point. In this paper the source points are positioned outside the physical domain, avoiding singularities. It is also presented the remainder operational procedures and numerical results to demonstrate the good performance of the Boundary Element Method, are also presented.

2. Integral Equation to Elastostatics

Is well-knwon the differential governing equation to elastostatic homogeneous isotropic linear problem (Timoshenko, 1980). This equation includes hypothesis such as Hooke's Law, small strains and equilibrium equation related to in non-deformated configuration of the solid body domain Ω. Governing equations are usually referred as Navier Equations, given by:

$$Gu_{j,kk} + \frac{G}{1-2\nu} u_{k,kj} = 0 \tag{1}$$

In previous equation G is the shear modulus and ν is the Poisson modulus. In completeness the former equation it is necessery to stablish essential and natural boundary conditions in $\Gamma = \Gamma_1 + \Gamma_2$, that is:

$$u_j = 0 \;\; in\, \Gamma_1 \; (essential) \tag{2}$$

$$p_j = 0 \;\; in\, \Gamma_2 \; (natural) \tag{3}$$

The inverse integral boundary equation associated to equation (1) can be obtained thrhough different approaches, such as residual methods. Here, the dedution of the inverse integral form starts from the following sentence:

$$G \int_\Omega u_{j,kk}\, u^*_{ij} d\Omega + \frac{G}{1-2\nu} \int_\Omega u_{k,kj}\, u^*_{jj} d\Omega = 0 \tag{4}$$

In the previous equation, $u*_{ij}$ is the diadic fundamental solution associated to correlated problem in which the domain is infinite and the external load is singular, usually named Kelvin fundamental solution (see Brebbia et al, 1984). After suitable mathematical operations, using specially integration by parts and the Divergence Theorem, it can be written:

$$C_{ij}(P) u_j(P) + \int_\Gamma p^*_{ij} u_j d\Gamma = \int_\Gamma u^*_{ij} p_j d\Gamma \tag{5}$$

It must be noticed that p^*_{ij} are fundamental tractions in diadic form and the $C_{ij}(P)$ are coefficients based to the relative position of source point "P" and the field point in the domain Ω (Brebbia, 1978). When constant elements are used in the boundary discretization the diadic coefficient $C_{ij}(P)$ generates a diagonal matricial. The same thing occurs when higher order elements are employed in smooth boundaries (Brebbia, 1979). In other situations $C_{ij}(P)$ needs to be determined by the rigid body displacement technique (Brebbia et al, 1984).

3. Axisymmetric Formulation

In according to the features of axisymmetric problems it is interesting to transform the cartesian coordinates system (x_1, y_1, z_1) to a cylindrical one (r,θ,z). Let the following relations:

$$x_1 = rcos(\theta), \;\; x_2 = rsin(\theta), \;\; x_3 = z \tag{6}$$

If $[x_1(P), x_2(P), x_3(P)]$ and $[x_1(Q), x_2(Q), x_3(Q)]$ are the cartesian coordinates of two arbitrary points "P" e "Q" , the cylindrical Euclidian distance is given by (Cisternas et all,1986):

$$R(P,Q) = \sqrt{r_i^2 + r_j^2 + 2 r_i r_j cos(\theta_j) + (z_i - z_j)^2} \tag{7}$$

Due to space limitations, it is not possible present the cartesian form of elastostatic Boundary Element integral equation, but it is trivial. Brebbia (1981) has more details about it. In cylindrical matrix form, the matrix integral equation is given by:

$$C(P) \begin{bmatrix} u_r(P) \\ u_\theta(P) \\ u_z(P) \end{bmatrix} + \sum_1^{NE} \int_{\Gamma^*} \begin{pmatrix} p^*_{rr} & p^*_{r\theta} & p^*_{rz} \\ p^*_{\theta r} & p^*_{\theta\theta} & p^*_{\theta z} \\ p^*_{zr} & p^*_{z\theta} & p^*_{zz} \end{pmatrix} \begin{bmatrix} u_r(Q) \\ u_\theta(Q) \\ u_z(Q) \end{bmatrix} 2\pi r_j d\Gamma^* =$$

$$= \sum_1^{NE} \int_{\Gamma^*} \begin{pmatrix} p^*_{rr} & p^*_{r\theta} & p^*_{rz} \\ p^*_{\theta r} & p^*_{\theta\theta} & p^*_{\theta z} \\ p^*_{zr} & p^*_{z\theta} & p^*_{zz} \end{pmatrix} \begin{bmatrix} p_r(Q) \\ p_\theta(Q) \\ p_z(Q) \end{bmatrix} 2\pi r_j d\Gamma^* \tag{8}$$

In the equation (8) NE represents the number of boundary elements. In a field point "X" on the boundary, the relations between cartesian and cylindrical displacements $\vec{u}(X)$ and tractions $\vec{p}(X)$ are expressed by:

$$\vec{u}(X) \begin{bmatrix} u_1(X) \\ u_2(X) \\ u_3(X) \end{bmatrix} = \frac{1}{2\pi} T(X) \vec{u}(X) = \frac{1}{2\pi} T(X) \begin{bmatrix} u_r(X) \\ u_\theta(X) \\ u_z(X) \end{bmatrix} \tag{9}$$

$$\vec{p}(X) \begin{bmatrix} p_1(X) \\ p_2(X) \\ p_3(X) \end{bmatrix} = \frac{1}{2\pi} T(X) \vec{p}(X) = \frac{1}{2\pi} T(X) \begin{bmatrix} p_r(X) \\ p_\theta(X) \\ p_z(X) \end{bmatrix} \tag{10}$$

Thus, considering the arbitrary angular coordinates of the source point "P" as zero, the transform tensors are given by:

$$T(Q) = \begin{pmatrix} cos\ \theta_j & -sen\ \theta_j & 0 \\ sin\ \theta_j & cos\ \theta_j & 0 \\ 0 & 0 & 1 \end{pmatrix},\ T(P) = \begin{pmatrix} cos\ 0 & -sen\ 0 & 0 \\ sin\ 0 & cos\ 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

Another important consideration is the substitution of infinitesimal cartesian three-dimensional boundary $d\Gamma$ by the infinitesimal revolution Γ^*, that is:

$$d\Gamma = 2\pi r_j d\Gamma^* \tag{12}$$

The components of fundamental displacement tensor are presented in equations (13) to (18). The intermediary steps were suppressed by convenience:

$$u^*_{rr} = \frac{1}{2\pi}\int_0^{2\pi}(u^*_{11}cos\ \theta_j + u^*_{12}sin\ \theta_j)d\theta_j =$$

$$= \frac{A}{cr_ir_j}\left\{\left[(3-4\nu)a + \hat{z}^2\right]K\left(m,\pi/2\right) - \left[(3-4\nu)c^2 + \frac{a\hat{z}^2}{d}\right]E\left(m,\pi/2\right)\right\} \tag{13}$$

$$u^*_{rz} = \frac{1}{2\pi}\int_0^{2\pi}u^*_{13}d\theta_j = \frac{A\hat{z}}{cr_i}\left[K\left(m,\pi/2\right) + \frac{f}{d}E\left(m,\pi/2\right)\right] \tag{14}$$

$$u^*_{zr} = \frac{1}{2\pi}\int_0^{2\pi}(u^*_{31}cos\ \theta_j + u^*_{32}sin\ \theta_j)d\theta_j = \frac{A\hat{z}}{cr_j}\left[-K\left(m,\pi/2\right) + \frac{e}{d}E\left(m,\pi/2\right)\right] \tag{15}$$

$$u^*_{zz} = \frac{1}{2\pi}\int_0^{2\pi}u_{33}d\theta_j = \frac{2A\hat{c}}{cr_j}\left[(3-4\nu)K\left(m,\pi/2\right) + \frac{\hat{z}^2}{d}E\left(m,\pi/2\right)\right] \tag{16}$$

$$u^*_{\theta\theta} = \frac{1}{2\pi}\int_0^{2\pi}(u^*_{22}cos\ \theta_j - u^*_{21}sin\ \theta_j)d\theta_j = \frac{4A(1-\nu)}{cr_ir_j}\left[aK\left(m,\pi/2\right) - c^2E\left(m,\pi/2\right)\right] \tag{17}$$

$$u^*_{r\theta} = u^*_{\theta r} = u^*_{z\theta} = u^*_{\theta z} = 0 \tag{18}$$

The components of fundamental traction tensor are presente:

$$p^*_{rr} = \mu n_z\left(\frac{\partial u^*_{rr}}{\partial z} + \frac{\partial u^*_{rz}}{\partial r}\right) + \frac{2\mu n_r}{(1-2\nu)}\left[(1-\nu)\frac{\partial u^*_{rr}}{\partial r} + \nu\left(\frac{u^*_{rr}}{r_j} + \frac{\partial u^*_{rz}}{\partial z}\right)\right] \tag{19}$$

$$p^*_{rz} = \mu n_r\left(\frac{\partial u^*_{rr}}{\partial z} + \frac{\partial u^*_{rz}}{\partial r}\right) + \frac{2\mu n_z}{(1-2\nu)}\left[(1-\nu)\frac{\partial u^*_{rz}}{\partial z} + \nu\left(\frac{u^*_{rr}}{r_j} + \frac{\partial u^*_{rr}}{\partial r}\right)\right] \tag{20}$$

$$p^*_{zr} = \mu n_z\left(\frac{\partial u^*_{zr}}{\partial z} + \frac{\partial u^*_{zz}}{\partial r}\right) + \frac{2\mu n_r}{(1-2\nu)}\left[(1-\nu)\frac{\partial u^*_{zr}}{\partial r} + \nu\left(\frac{u^*_{zr}}{r_j} + \frac{\partial u^*_{zz}}{\partial z}\right)\right] \tag{21}$$

$$p^*_{zz} = \mu n_r\left(\frac{\partial u^*_{zr}}{\partial z} + \frac{\partial u^*_{zz}}{\partial r}\right) + \frac{2\mu n_z}{(1-2\nu)}\left[(1-\nu)\frac{\partial u^*_{zz}}{\partial z} + \nu\left(\frac{u^*_{zr}}{r_j} + \frac{\partial u^*_{zr}}{\partial r}\right)\right] \tag{22}$$

$$p^*_{\theta\theta} = \mu n_z\left(\frac{\partial u^*_{\theta\theta}}{\partial z}\right) + \mu n_r\left(\frac{\partial u^*_{\theta\theta}}{\partial r} - \frac{u^*_{\theta\theta}}{r_j}\right) \tag{23}$$

$$p^*_{r\theta} = p^*_{\theta r} = p^*_{z\theta} = p^*_{\theta z} = 0 \tag{24}$$

The fundamental stress tensor is more complex. Its depends of the displacement derivatives of fundamental displacements or the strain tensor. Such derivatives are presented below. The intemediary steps were suppressed by convenience:

$$\frac{\partial u^*_{zr}}{\partial r} = \frac{A}{cr_ir_j}\left[\frac{\hat{z}^2}{2r_j}\left(\frac{ae}{dc^2} - 3\right) - (3-4\nu)\frac{r_i^2 + \hat{z}^2}{r_j}\right]K\left(m,\pi/2\right) +$$

$$+\frac{A}{cr_ir_j}\left\{\frac{(3-4\nu)}{2r_j}\left(\frac{ae}{d} + c^2\right) + \frac{\hat{z}^2}{d}\left[e\left(\frac{r_i}{c^2} + \frac{1}{r_j}\right) + a\left(\frac{r_i + r_j}{c^2} - 2\frac{r_i - r_j}{d}\right)\right]\right\}E\left(m,\pi/2\right) \tag{25}$$

$$\frac{\partial u^*_{rr}}{\partial z} = \frac{A\hat{z}}{cr_ir_jd}\left[\frac{A\hat{z}^2}{c^2} - (5-4\nu)d\right]K(m,\pi/2) +$$

$$+\frac{A\hat{z}}{cr_ir_jd}\left[(5-4\nu)a - 2a\hat{z}^2\left(\frac{1}{c^2}+\frac{1}{d}\right) + 3\hat{z}^2\right]E(m,\pi/2) \tag{26}$$

$$\frac{\partial u^*_{rz}}{\partial r} = -\frac{A\hat{z}}{2cr_ir_jd}\left(\frac{ef}{c^2}+d\right)K(m,\pi/2) +$$

$$+\frac{A\hat{z}}{2cr_ir_jd}\left\{e\left[2f\left(\frac{1}{c^2}+\frac{1}{d}\right)+1\right] - 3f - 4r_j^2\right\}E(m,\pi/2) \tag{27}$$

$$\frac{\partial u^*_{rz}}{\partial z} = -\frac{A}{cr_id}\left[\left(\frac{\hat{z}^2 f}{c^2}+d\right)K(m,\pi/2)\right] + \frac{A}{cr_id}\left\{\hat{z}^2\left[3+2f\left(\frac{1}{c^2}+\frac{1}{d}\right)\right] - f\right\}E(m,\pi/2) \tag{28}$$

$$\frac{\partial u^*_{zr}}{\partial r} = A\left\{\frac{\hat{z}}{2cr^3r_j^2}\left(3c^2 - \frac{e^2}{d}\right)K(m,\pi/2) + \frac{\hat{z}}{cr_jd}\left[e\left(\frac{e}{c^2r_j} - \frac{2}{r_j} + 2\frac{r_i-r_j}{d}\right) - 2r_j\right]E(m,\pi/2)\right\} \tag{29}$$

$$\frac{\partial u^*_{zr}}{\partial z} = \frac{A}{cr_jd}\left(d - \frac{\hat{z}^2 e}{c^2}\right)K(m,\pi/2) + \frac{A}{cr_jd}\left\{\hat{z}^2\left[-3+2e\left(\frac{1}{c^2}+\frac{1}{d}\right)\right] - e\right\}E(m,\pi/2) \tag{30}$$

$$\frac{\partial u^*_{zz}}{\partial r} = -\frac{A}{cr_jd}\left[(3-4\nu)d\frac{\hat{z}^2 e}{c^2}\right]K(m,\pi/2) +$$

$$+\frac{A}{cr_jd}\left[(3-4\nu)e + \hat{z}^2\left(\frac{d}{c^2} + 4r_j\frac{r_i-r_j}{d} - 4\frac{r_j}{c^2}\right)\right]E(m,\pi/2) \tag{31}$$

$$\frac{\partial u^*_{zz}}{\partial z} = -\frac{2A\hat{z}^3}{dc^3}K(m,\pi/2) + \frac{2\hat{z}}{cd}\left[1-4\nu+2\hat{z}^2\left(\frac{1}{c^2}+\frac{1}{d}\right)\right]E(m,\pi/2) \tag{32}$$

$$\frac{\partial u^*_{\theta\theta}}{\partial r} = \frac{4A(\nu-1)}{cr_ir_j^2d}\left[8r_i^2r_j^2 + 4r_ir_j^3 + (-6r_ir_j - r_j^2 + c^2)c^2\right]K(m,\pi/2) +$$

$$+\frac{4A(\nu-1)}{cr_ir_j^2d}\left[-2r_i^2r_j^2 - 2r_ir_j^3 + (4r_ir_j + r_j^2 - c^2)c^2\right]E(m,\pi/2) \tag{33}$$

$$\frac{\partial u^*_{\theta\theta}}{\partial z} = \frac{4A\hat{z}(\nu-1)}{cr_ir_j}\left(K(m,\pi/2) - \frac{a}{d}E(m,\pi/2)\right) \tag{34}$$

$K(m,\pi/2)$ and $E(m,\pi/2)$ are complete elliptic integrais of first and second kind respectly, with modulus m^2; n_r, n_z, and n_θ are components of unitary normal boundary vector at "P". It is necessary to define the following variables:

$$\hat{z} = z_i - z_j \tag{35}$$

$$a = r_i^2 + r_j^2 + \hat{z}^2 \tag{36}$$

$$c = \sqrt{(r_i+r_j)^2 + \hat{z}^2} \tag{37}$$

$$d = (r_i - r_j)^2 + \hat{z}^2 \tag{38}$$

$$e = a - 2r_j^2 \tag{39}$$

$$f = e - 2\hat{z}^2 \tag{40}$$

$$m = 2\sqrt{(r_ir_j)/c} \tag{41}$$

$$A = \frac{1}{16\pi^2\mu(1-\nu)} \tag{42}$$

4. Discretization Procedure

The discretization procedure consists of the next step with the Boundary Element Method approach. It means, in general sense, that the Γ^* is divided into a finite number of elements in which displacements and tractions are approximated. In each boundary element, nodal points are chosen to represent the whole geometry and physical field variation. This work employed quadratic shape functions to simulated the behavior of displacements and tractions.

It is necessary to generate a system of equations to solve the matrix equation (8). This is done considering different *P* source points, such as the number of source points is equal to field *Q* points or nodal points. The vectorial kind of physical variables implies that three equations need to generated to each nodal point. As mentioned earlier, hard solving singular integrals would need to be solved using traditional collocation of source points on the boundary. Here the positions of source points are external to the physical domain, avoiding singularities. It means that:

$$C(P) \equiv 0 \tag{43}$$

About the position of external source point, it followed the strategy employed by Fernandes & Venturini (2002). These authors suggest the use of d_i like a suitable distance between nodal points and the most near source points choosen

$$d_i = a_i l_j \tag{44}$$

In the equation (44) l_j is the average size of adjacent boundary elements and is an arbitrary parameter betwen 0.1 and 0.5.

5. Aplications

5.1 Hollow Sphere

Figure (1) illustrates a hollow sphere with internal radius "a" and external radius "b", submitted only to an internal unitary traction and null external load.

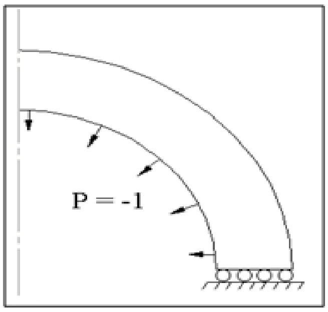

Figure 1. Boundary conditions in a hollow sphere

A half sphere may represent the problem since the displacement in *z* axis is restricted. The analythical solution of this problem is given by:

$$\sigma_{\theta\theta} = \frac{a^3}{b^3 - a^3}\left(1 + \frac{b^3}{2r^3}\right) \tag{45}$$

The mesh of boundary elements used eight quadratic elements. Each inner and external boundaries were employed two elements. The rest was used in the discretization of the horizontal support. To implement the computational simulation, the following data was used:

$$a = 1,\ \ b = 2m;\ \nu = 0\ \ and\ \ E = 1,0\ Pa. \tag{46}$$

The following tables present the comparison between analytical solution and numerical one.

Table 1. Radial displacement $u_r(m)$ in radial direction

Radius(m)	BEM	Analytical	Error %
1.125	-0.6125	-0.6122	0.04
1.375	-0.4983	-0.4987	0.07
1.625	-0.4480	-0.4485	0.12
1.875	-0.4297	-0.4304	0.16

Table 2. Normal Stress $\sigma_{\theta\theta}$(Pa) in radial direction

Radius(m)	BEM	Analytical	Error %
1.125	0.5429	0.5442	0.24
1.375	0.3606	0.3627	0.58
1.625	0.2746	0.2760	0.52
1.875	0.2302	0.2295	0.29

5.2 Hollow Cylinder

A hollow cylinder is now analysed with internal radius "a" and external radius "b", submited to a unitary shear stress in its internal part, like illustrated in figure (2).

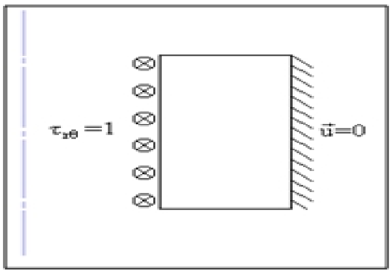

Figure 2. Boundary conditions for the revolution section of hollow cylinder

The analytical solutions of shear stress and angular displacement, are:

$$\tau_{r\theta} = \frac{a^2}{r^2} \quad (47)$$

$$u_\theta = \frac{a^2}{2G}\left(\frac{r}{b^2} - \frac{1}{r}\right) \quad (48)$$

Eight quadratic boundary elements were employed. The following data were adopted :

$$a = 0,1m, \;\; b = 1m \;\; and \;\; G = 0,5\;Pa. \quad (49)$$

The following tables presents the comparison between analythical solution and numerical one.

Table 3. Angular displacement in radial direction u_θ(m)

Radius(m)	BEM	Analytical	Error %
0.10	0.0988	0.0990	0.23
0.25	0.0377	0.0375	0.66
0.55	0.0126	0.0127	0.91
0.85	0.0032	0.0033	1.12

Table 4. Shear stress $\tau_{r\theta}$(Pa) to three points on the external boundary

Point	BEM	Analytical	Error %
1°	0.010005	0.01	0.05
2°	0.009998	0.01	0.02
3°	0.010005	0.01	0.05
Average	**0.010000**	**0.01**	**0.04**

6. Conclusions

The Boundary Element Method is a very efficient technique to solve axisymmetric problems. In this class of problems, the economy with BEM in processing computational time is meaningful, due to the reduced number of nodal points to discretize the domain. In the numerical model the easiness to input data is also an other very important BEM feature. The very high accuracy of its results is another important aspect. Engineers and other professionals had very significants advantages to solve axisymmetric problems using a computational code with boundary element procedures. However, the implementation of the computational code of BEM is more complex. The most difficult aspect is related to the elliptic integrals obtained with the integration of the three-dimensional fundamental solution onto the angular coordinates. Both analytical and numerical procedures to solve theses integral are very cumbersome.

This work used a very interesting procedure to reduce the difficulties of elastic axisymmetric boundary element models. The source points were positioned outside of the boundary, that is, externally to the physical domain to avoid singular integrations when the source and field points are coincidents. On the other hand, external source points were choosen according with a well-known empirical rule in which the suitable distance of field points is prescribed to avoid inaccuracy. This procedure also eliminates another important difficulty with higher order elements in axisymmetric analysis: the C_{ij} coefficient is not simple to determinate in this class of problems, due to the impossibility to employ the rigid body technique, since the radial direction does not allow displacements. With the external position of source point, the C_{ij} coefficient is null.

The numerical results were very good, as presented in the previous examples. The average percentual error obtained was below 1%, despite the few boundary elements used in discretization. The refinement of the mesh is presented in this paper for space limitations, but the convergence of numerical and analytical results also was verified with numerical simulations implemented.

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