HIERARCHICAL METHODS APPLIED TO FREE VIBRATION OF EULER-BERNOULLI BEAMS

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Abstract. *This work deals with hierarchical finite element methods applied to free vibration of Euler-Bernoulli beams. The variational problem of free vibration is formulated and the main aspects of a hierarchical refinement are discussed. The conventional hierarchical p-version of FEM, the Composite Element Method (CEM) and a trigonometric version are presented. The CEM is developed by enrichment of the conventional FEM local solution space with non-polynomial functions obtained from closed form solutions of classical theory. This approach results in a hierarchical refinement called c-version. The trigonometric version is obtained combining trigonometric functions with the conventional FEM solution space. Automatic refinements without recalculate stiffness and mass coefficients can be obtained with these hierarchical versions. The application of hierarchical methods in vibration analysis of beams is investigated. The eigenvalues obtained by hierarchical methods are compared with those obtained by analytical solution and by h-version of FEM. The numerical results shows that the hierarchical refinements presented in this work have in almost all cases higher rates of convergence than those obtained by the h-version of FEM. In the other cases the rates of convergence obtained were equal the h-version rates.*
Keywords*: hierarchical methods, free vibration, vibration analysis, finite element method*

1. Introduction

The recent great development of structural engineering, material science and construction techniques have resulted in tall and slender structures. In these cases, it is essential to know the behavior of the structure over dynamical effects. Nowadays many researchers have developed vibration analysis methods.

The frequencies and vibration mode shapes of a structure are obtained from the solution of free vibration problems. These vibration modes can be used to represent the structural displacement when dynamical loads are applied (forced vibration).

Real structures are continuous systems that have infinite vibration frequencies but in approximated methods just a finite number of frequencies related with the degrees of freedom are obtained. When a poor discratization is used the precision of the lower frequencies is good, but the precision decrease for higher frequencies.

For some applications just some lower frequencies are needed. In structures loaded by motors working in high frequencies it is important to know precisely the higher vibration frequencies of the structure. In these cases the solution methods and models must allow successive refinements of solution until obtain the desired precision.

In the Finite Element Method (FEM), the approximated solution can be improved using refinement techniques. In conventional FEM two types of approach are used: *h* and *p*-versions. The *h*-version consists of the refinement of element mesh. Recent works (Ribeiro, 2001) (Campion and Jarvis, 1996) define *p*-version as the increase in the number of form functions in the element without change in the mesh. For polynomials form functions, the *p*-version consists of increasing the polynomial degree in the solution. Some researchers (Ganesan and Engels, 1991) (Zeng, 1998a) (Ribeiro, 2001) have used non-polynomial form functions to refine FEM solutions.

This work presents the variational form to the problem of free vibration of Euler-Bernoulli beams and analyses the application of some hierarchical refinement methods in the solution of these problems.

2. Variational form of the free vibration of Euler-Bernoulli beams

The Euler-Bernoulli beam consists of a straight beam with lateral strain (Fig. 1). The basic hypotheses are (Craig, 1981): (a) There is a neutral line (*x*-axis) when there is no traction nor compression; (b) The cross sections which are straight and normal to the neutral line before deformation remain straight and normal after deformation; (c) The material is elastic, linear and homogeneous; (d) Normal stress σ_y and σ_z are too small when compared to the axial stress

σ_x and, for this reason, they are neglected; (e) The *xy*-plane is a principal plane; and (f) The rotational inertia of the beam is not considered.

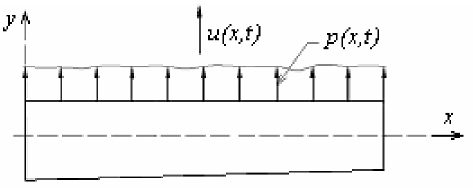

Figure 1. Straight Euler-Bernoulli beam.

The vibration of the Euler-Bernoulli beam is a time dependent problem. The movement equation that governs this problem is a partial differential equation. The problem consists of find the lateral displacement $\overline{u} = \overline{u}(x,t)$ which satisfies

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 \overline{u}}{\partial x^2} \right) + \rho A \frac{\partial^2 \overline{u}}{\partial t^2} = p(x,t) \qquad (1)$$

where A is the cross section area, I is the transverse moment of inertia, E is the Young modulus, ρ is the specific mass, p is the load and t is the time. The solution $\overline{u} = \overline{u}(x,t)$ must satisfy boundary and initial conditions defined in the problem.

According to Carey and Oden (1984b), the most popular form to obtain the variational form of a time dependent problem is consider the time t like a real parameter and develop a family of variationals problems in t. This consists in select test functions $v = v(x)$, independent of t, and apply the weighted-residual method. If the finite element method is used to represent the spatial behavior of the solution, one obtains a system of ordinary differential equations of the degrees of freedom as functions of the parameter t. This approach is called semidiscrete formulation of the problem.

Using the weighted-residual method, according to Carey and Oden (1984b), to develop an integral statement of Eq.(1), the solution $\overline{u} = \overline{u}(x,t)$ must satisfy

$$\int_0^L \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 \overline{u}}{\partial x^2} \right).v.dx + \int_0^L \rho A \frac{\partial^2 \overline{u}}{\partial t^2}.v.dx = \int_0^L p(x,t).v.dx \qquad (2)$$

for admissible test functions $v = v(x)$ at any time $t \in (0,T]$.

Integrating Eq.(2) by parts twice, one obtains:

$$\left[v.\frac{\partial}{\partial x}\left(EI \frac{\partial^2 \overline{u}}{\partial x^2} \right) \right]_0^L - \left[\frac{\partial v}{\partial x}.\left(EI \frac{\partial^2 \overline{u}}{\partial x^2} \right) \right]_0^L + \int_0^L \left(EI \frac{\partial^2 \overline{u}}{\partial x^2} \right).\frac{\partial^2 v}{\partial x^2}.dx + \int_0^L \rho.A \frac{\partial^2 \overline{u}}{\partial t^2}.v.dx = \int_0^L p(x,t).v.dx \qquad (3)$$

It is necessary to introduce the boundary and initial conditions to complete the problem. Analyzing first the boundary $x = 0$, the common boundary conditions for beams are:

Cantilever end: $\overline{u}(0,t) = 0$ and $\left. \frac{\partial \overline{u}}{\partial x} \right|_{x=0} = 0$ (4)

Simple supported end: $\overline{u}(0,t) = 0$ and $M(0,t) = \left. \left(EI \frac{\partial^2 \overline{u}}{\partial x^2} \right) \right|_{x=0} = 0$ (5)

Free end: $$M(0,t) = \left(EI \frac{\partial^2 \overline{u}}{\partial x^2} \right)\bigg|_{x=0} = 0 \text{ and } Q(0,t) = \frac{\partial}{\partial x}\left(EI \frac{\partial^2 \overline{u}}{\partial x^2} \right)\bigg|_{x=0} = 0 \qquad (6)$$

where M is the bending moment and Q is the shear force. These boundary conditions are also applicable to boundary $x = L$.

The admissible test functions $v = v(x)$ must satisfy the same boundary conditions of the solution $\overline{u}(x,t)$. To any combination of boundary conditions (Eqs. (4), (5) and (6)) in boundaries $x = 0$ and $x = L$, the Eq.(3) becomes:

$$\int_0^L \left(EI \frac{\partial^2 \overline{u}}{\partial x^2} \right) . \frac{\partial^2 v}{\partial x^2} . dx + \int_0^L \rho . A \frac{\partial^2 \overline{u}}{\partial t^2} . v . dx = \int_0^L p(x,t) . v . dx \qquad (7)$$

Particularizing the problem to the case of free vibration of a uniform straight beam, where *E*, *A*, *I* and ρ are constants and $p(x,t) = 0$, the Eq. (7) becomes:

$$EI \int_0^L \frac{\partial^2 \overline{u}}{\partial x^2} . \frac{\partial^2 v}{\partial x^2} . dx + \rho . A \int_0^L \frac{\partial^2 \overline{u}}{\partial t^2} . v . dx = 0 \qquad (8)$$

According to Carey and Oden (1984b), in vibration problems, one assumes periodic solutions $\overline{u}(x,t) = e^{i\omega t} u(x)$, where ω is the vibration frequency. The free vibration of a uniform Euler-Bernoulli beam becomes in an eigenvalue problem with variational statement: find a pair (λ, u), with $u \in H^2(0,L)$ and $\lambda \in \mathbf{R}$, so that

$$EI \int_0^L \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} . dx - \rho A \lambda \int_0^L u v dx = 0 \qquad (9)$$

for admissible test functions $v \in H^2(0,L)$, where $\lambda = \omega^2$.The variational statement of this eigenvalue problem can also be written as: find (λ, u), with $u \in H^2(0,L)$ and $\lambda \in \mathbf{R}$, so that

$$B(u,v) = \lambda .(u,v) \qquad (10)$$

for all admissible test functions $v \in H^2(0,L)$, where $B(u,v)$ is the bilinear form and (u,v) the scalar product in L^2 space, obtained from

$$B(u,v) = \frac{EI}{\rho A} \int_0^L \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} . dx \qquad (11)$$

$$(u,v) = \int_0^L u v dx . \qquad (12)$$

3. Finite element method

The approximation by finite elements of the free vibration problem of an Euler-Bernoulli beam consists to rewrite the variational form of the problem (Eq. (10)) in an approximated subspace $H^h \subset H^2(0,L)$. The approximated eigenvalue problem in Eq. (10) becomes: find $\lambda_h \in \mathbf{R}$ and $u_h \in H^h(0,L)$ so that

$$B(u_h, v_h) = \lambda_h .(u_h, v_h) \qquad \forall v_h \in H^h \qquad (13)$$

The approximated solution $u_h(x)$ from finite elements can be written, to discretization in *N* nodes, in the following form:

$$u_h(x) = \sum_{j=1}^{N} u_j \phi_j(x) \tag{14}$$

where $\{\phi_j\}$ are the global base functions of the subspace H^h and $\{u_j\}$ are the degrees of freedom. Replacing the Eq. (14) in Eq.(13) and taking $v_h = \phi_i$, $i = 1,2,\ldots,N$, one obtains:

$$\sum_{j=1}^{N}\left(EI\int_0^L \frac{\partial^2\phi_i}{\partial x^2}\frac{\partial^2\phi_j}{\partial x^2}dx \right).u_j = \lambda_h \sum_{j=1}^{N}\left(\rho A\int_0^L \phi_i\phi_j dx \right).u_j \ , \qquad i = 1,2,\ldots,N \tag{15}$$

or, in matrix form,

$$\mathbf{Ku} = \lambda_h \mathbf{Mu} \tag{16}$$

where **K** is called stiffness matrix and **M** is called mass matrix, and they are defined in Eq. (15). The Equation (15) corresponds to a generalized eigenvalue problem where λ_h are the eigenvalues related to the vibration frequencies ω and the vectors **u** are the eigenvectors related to the vibration shape modes of the beam.

After the discratization of the problem domain Ω in sub domains Ω_e, called elements, it is necessary determinate the contribution of each element to the elementary stiffness matrix coefficients (k_{ij}^e) and elementary mass matrix coefficients (m_{ij}^e), obtained by:

$$k_{ij}^e = EI\int_{\Omega_e} \frac{\partial^2\psi_i^e}{\partial x^2}\frac{\partial^2\psi_j^e}{\partial x^2}dx \tag{17}$$

$$m_{ij}^e = \rho A\int_{\Omega_e} \psi_i^e \psi_j^e dx \tag{18}$$

where the local form function ψ_i^e is the restriction of the base function ϕ_i in element Ω_e.

The adequated choice of the local form functions ψ_i^e determinates different solution methods with distinct features and rates of convergence. The next topics present form functions sets that create hierarchical methods. In general, according to Reddy (1986), the form functions are developed to master elements and then they are mapped to the real elements obtained from the finite element mesh. In this work the master element domain is Ω_e (0,1).

4. Hierarchical methods

According to Ribeiro (2001), Zienkiewicz *et al.* (1982) and, Carey and Oden (1984a,b), in a *p* refinement, if the set of form functions of an approximation of order *p* constitutes a subset of the set of form functions of an approximation of order *p+1*, this refinement is called hierarchical. The hierarchical form functions were introduced by Zienkiewicz, Irons, Scott and Campbell in 1971, according to the work of Zienkiewicz *et al.* (1982).

Campion and Jarvis (1996) presents the major features of the hierarchical methods: the retention of the stiffness matrix coefficients as the order of the interpolation is increased and the high convergence rates without the need for mesh refinement.

The application of hierarchical methods in the solution of structural vibration problems allows that stiffness and mass matrices previously calculated can be maintained and just the matrices coefficients related to the new form function must be calculated. This feature reduces the computational effort necessary to build the matrices in each step of the refinement.

It is important to stand out that the new degrees of freedom introduced by the adding of new form functions in a hierarchical refinement do not corresponds to the physical quantities generally related to the elementary degrees of freedom. Just the degrees of freedom that define the mesh stay related to the physical quantities of interest, which can correspond, for instance, to the nodal displacements. However, these physical quantities can be easily obtained replacing the point coordinates in the interpolation function.

Ribeiro (2001) shows that high order polynomials are ill conditioned. Some other researchers have used trigonometric functions to interpolate the displacements in structural vibration problems.

4.1. Conventional hierarchical *p* refinement

The conventional *p* refinement uses polynomials form functions. Taking the uniform beam element (Fig.2) with two degrees of freedom per each node, the approximated solution in the element domain can be defined as:

$$u_h^e(\xi) = \psi_1^e(\xi).u_1 + \psi_2^e(\xi).\theta_1 + \psi_3^e(\xi).u_2 + \psi_4^e(\xi).\theta_2 \tag{19}$$

or in matrix form: $u_h^e(\xi) = \mathbf{N}^T\mathbf{q}$ (20)

where $\xi = \dfrac{x}{L}$, $\mathbf{N}^T = [\psi_1^e \quad \psi_2^e \quad \psi_3^e \quad \psi_4^e]$ and $\mathbf{q}^T = [u_1 \quad \theta_1 \quad u_2 \quad \theta_2]$, L is the element length, u_1 and u_2 are the nodal displacements, and, θ_1 and θ_2 are the nodal rotations.

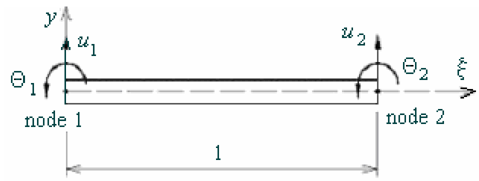

Figure 2. Euler-Bernoulli beam element.

Using Hermitian polynomials as local form functions obtained from Lagrangian polynomials by the method described in the work of Augarve (1998), one obtains:

$$\mathbf{N}^T = \left[1 - 3.\xi^2 + 2.\xi^3 \quad L.\left(\xi - 2.\xi^2 + \xi^3\right) \quad 3.\xi^2 - 2.\xi^3 \quad L.\left(\xi^3 - \xi^2\right)\right] \tag{21}$$

The hierarchical *p* refinement can be obtained adding a new node between the original element nodes. The local form functions in Eq. (21) are maintained and just the new local form functions of superior order related to the new node are added. By this approach the new form function in a three node element obtained from two node element are:

$$\psi_5^e = 16\xi^4 - 32\xi^3 + 16\xi^2 \tag{22}$$

$$\psi_6^e = L(16\xi^5 - 40\xi^4 + 32\xi^3 - 8\xi^2) \tag{23}$$

Two new degrees of freedom u_3 and θ_3 appear and they do not correspond to the nodal displacement and rotation. Adding new nodes in the element domain and taking the previous form functions, one obtains new local bases hierarchicaly superiors. Taylor, Zienkiewicz and Oñate (1998) present a hierarchical finite element method based in the unity partition.

4.2. Composite element method (CEM)

A hierarchical *p* refinement can also be obtained using a FEM conventional element with local form functions enriched by adding non-polynomial functions related to the closed form solutions from classical theory.

Weaver Junior and Loh (1985) used analytical solution as form functions to lateral displacements in the analysis of local vibration mode shapes of trusses. After this approach was applied by Ganesan and Engels (1991) to obtain a hierarchical model of finite elements of Euler-Bernoulli beams. Zeng (1998b) developed elements of trusses, Euler-

Bernoulli beams and frames using this approach to vibration analysis. In the work of Zeng (1998a, b) this technique was called Composite Element Method (CEM).

According to CEM, the approximated solution in the element domain of an Euler-Bernoulli beam is obtained by:

$$u_h^e = \mathbf{N}^T \mathbf{q} + \mathbf{Ø}^T.\mathbf{c} \tag{24}$$

where $\mathbf{q}^T = [u_1 \quad \theta_1 \quad u_2 \quad \theta_2]$, the vector **N** contains the FEM form functions obtained in Eq.(21) and the vectors **Ø** and **c** are obtained by:

$$\mathbf{Ø}^{\mathrm{T}}(\xi) = [F_1 \quad F_2 \quad \dots \quad F_n] \tag{25}$$

$$\mathbf{c}^T = [c_1 \quad c_2 \quad \cdots \quad c_n] \tag{26}$$

$$F_r = \mathrm{sen}(\lambda_r.\xi) - \mathrm{senh}(\lambda_r.\xi) - \frac{\mathrm{sen}\,\lambda_r - \mathrm{senh}\,\lambda_r}{\cos\lambda_r - \cosh\lambda_r}.[\cos(\lambda_r.\xi) - \cosh(\lambda_r.\xi)] \tag{27}$$

where c_i are the coefficients that multiply the analytical solutions F_r obtained from the solution of the free vibration problem of an Euler-Bernoulli beam with all end displacements contained, for $r = 1,2,\dots$.

The new degrees of freedom obtained from the form functions enrichment do not have direct physical meaning and they were called *c* degrees of freedom by Zeng (1998b). The enrichment proposed by CEM produce hierarchical models and better results than those obtained from *h*-version of FEM. The hierarchical refinement produced when the number of analytical functions in the approximated solution is increased was called *c* refinement by Zeng (1998a, b).

4.3. Trigonometric refinement

Some researchers have been proposed the use of trigonometric functions to enrichment of the base of FEM form functions. In this case, an approximated solution in the element domain of the beam is obtained by Eq. (24) using appropriate trigonometric functions to substitute the analytical functions of CEM.

Ganesan and Engels (1991) proposed to use trigonometric functions in the form

$$F_r = \cos[(r-1)\pi\xi] - \cos[(r+1)\pi\xi], \qquad r = 1,2,\dots \tag{28}$$

in vibration analysis of Euler-Bernoulli beams.

Ribeiro (2001) uses trigonometric base functions in the form

$$F_r = \mathrm{sen}(r\pi\xi)\,\mathrm{sen}(\pi\xi), \qquad r = 1,2,\dots \tag{29}$$

in geometrically non linear vibration analysis of plane beams and frames by finite elements.

Analyzing the Eqs. (28) and (29) one observes that the functions are equivalent with different amplitudes and they results in directly proportional stiffness matrices. Both forms results in hierarchical methods.

5. Applications

For the numerical verification of the presented hierarchical methods, the free lateral vibration of a simply supported beam (Fig. 3) proposed by Ganesan and Engels (1991), with length *L=1,8 m*, elasticity modulus *E=210 GPa*, density ρ*=8000 kg/m³*, cross section area $A=0{,}7 \times 10^{-3}\ m^2$ and moment of inertia $I=0{,}035 \times 10^{-6}\ cm^4$, is analyzed.

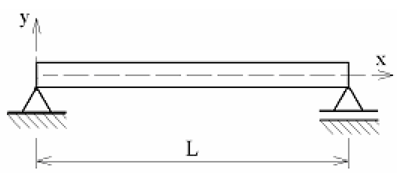

Figure 3. Simply supported Euler-Bernoulli beam.

The exact equation for the natural frequencies (ω) of the simply supported Euler-Bernoulli beam is (Craig,1981):

$$\omega = \left(\frac{i\pi}{L}\right)^2 \sqrt{\left(\frac{EI}{\rho A}\right)}, \qquad i = 1,2,\ldots \tag{30}$$

For this application the approximated eigenvalues λ_h were obtained by FEM using the hierarchical refinements presented in this work. The results were also compared to those obtained from the *h*-version of FEM taking a regular subdivision of the mesh. In all these hierarchical refinements, the beam was described geometrically by one element and the successively refinements were obtained increasing the number of local form functions. In the trigonometric refinement, the form functions described in Eq. (28) were used.

Figures 4 and 5 present the evolution of relative error for the four earliest eigenvalues of the proposed problem as function of the total number of degrees of freedom used in each method. The relative error is presented in logarithmic scale and calculated by:

$$error = 100\frac{\lambda_h - \lambda}{\lambda} \tag{31}$$

where λ_h is the approximated eigenvalue and $\lambda = \omega^2$ is the exact eigenvalue obtained by Eq. (30).

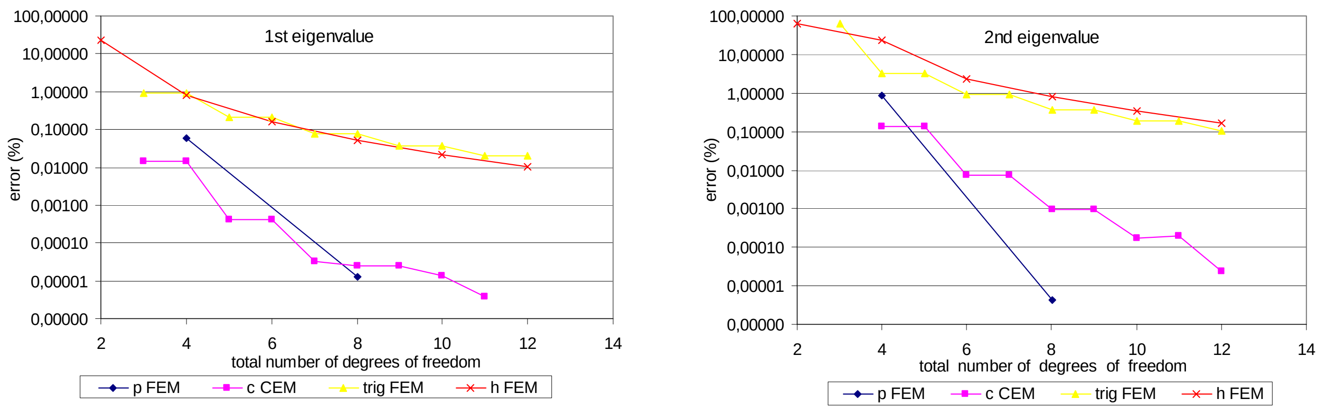

Figure 4. Relative error (%) for the 1st and 2nd beam eigenvalues.

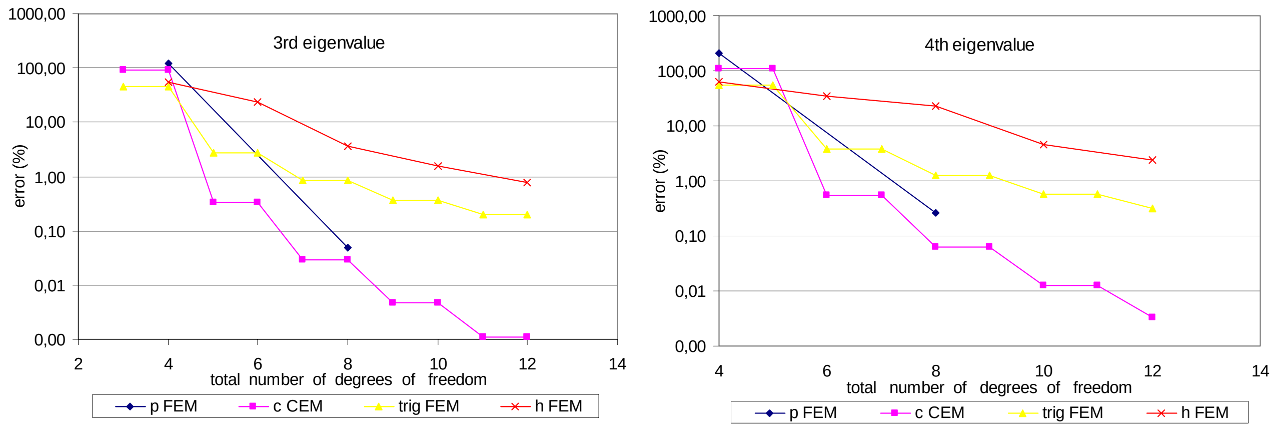

Figure 5. Relative error (%) for the 3rd and 4th beam eigenvalues.

Analyzing the results obtained for the simply supported beam, one observes that the CEM and the hierarchical *p* refinement have the greatest convergence rates. Nevertheless, like presented by Ribeiro (2001), high order polynomials are ill conditioned. In this work one verifies that when a 9 node hierarchical element was used, for the simply supported beam, spurious eigenvalues appeared. The cause of this phenomenon needs special investigation.

The CEM and the hierarchical *p* refinement presented better results than those obtained from *h* and trigonometric refinements, in special for higher eigenvalues. The trigonometric refinement presents convergence rates near the convergence rates of the *h* refinement. However, the results of the trigonometric refinement were more accurate than those obtained by the *h* refinement for higher eigenvalues.

6. Conclusion

This work presents the variational form to the free vibration problem of the uniform straight Euler-Bernoulli beam with common boundary conditions. The presented procedure allows obtaining the variational form for the free vibration of non uniform beams or with different boundary conditions, and for the forced vibration of Euler-Bernoulli beams.

Different hierarchical methods were presented. All of them utilize finite elements to solve free vibration problems of beams. These methods can be applied in other problems as, for example, structural instability problems.

In the Composite Element Method (CEM), the local form functions of a FEM conventional element are enriched by adding non-polynomial functions obtained from closed form solutions (exact) of the classical theory. This approach produces a hierarchical refinement called *c* refinement.

Another presented technique uses trigonometric form functions in the enrichment of the FEM form functions base. These refinements are called trigonometric refinements.

To compare the hierarchical methods, the eigenvalues of free vibration of a 1,8 m length simply supported beam were calculated. The analytical solution of this problem is well-known. The hierarchical methods were also compared to *h* refinement of FEM.

The results have shown that the hierarchical *p* refinement and the CEM present convergence rates greater than those obtained from trigonometric and *h* refinements. The trigonometric refinement presented convergence rates close to the rates of the *h*-version. Nevertheless, for higher eigenvalues, the results of the trigonometric refinement were better than those obtained from the *h* refinement.

We can get the conclusion, therefore, that presented hierarchical refinements showed convergence rates higher, or at least equal, than convergence rates obtained from *h* refinement. The CEM and the hierarchical *p* refinement presented the best results for the studied problem. The ill conditioning of the solution obtained from the *p* refinement with high order polynomials needs special investigation. Error estimates for the CEM and the trigonometric refinement also needs to be researched.

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