

USING SINGULAR AND NON-SINGULAR ELEMENTS AND BEM AND FEM STRATEGIES FOR THE COMPUTATION OF K-I IN LEFM

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Abstract. *In experimental mechanics, the determination of the stress intensity factor (K) is expensive and time consuming, but the knowledge of K, in Linear Elastic Fracture Mechanics (LEFM), is helpful for the examination of crack propagation in structures like pipelines and pressure vessels. An alternative to experimental determination of K is the numerical computation of K using numerical methods like the Finite Element Method (FEM) and the Boundary Element Method (BEM). However, stress gradients, in the area close to the crack tip, are difficult to be represented. For both, FEM and BEM, special elements are employed to represent the singularities idealized at the crack tip. This work presents several numerical strategies for the determinations of K using the FEM and BEM schemes with common elements without singularity, and with special elements with singularity at the tip of the crack. It is noticed that the singularities at the tip of the crack are mathematical idealizations and maybe are not very important for the calculation of K in practical engineering. This issue is discussed in this article. Several classic examples for the determination of K are presented and the results compared to values available in the literature.*

Keywords: *Fracture Mechanics, Finite Element, Boundary Element, Stress Intensity Factor.*

1. Introduction

Crack propagation is of great interest for the evaluation of structural integrity in pressure vessels, pipelines, aircraft fuselages, concrete dams, and other structures. Crack propagation can be evaluated using the Linear Elastic Fracture Mechanics (LEFM) parameters such as the Stress Intensity Factor (K) and the J-Integral (J), (Broek, 1989). Whilst K can be found experimentally, it is easier to obtain this parameter numerically. Numerical methods provide a quick, accurate and low cost alternative for the calculation of K. Amongst the numerical methods used to this end is Finite Element Method (FEM), which is straightforward, versatile and widely used in all fields of engineering and more recently Boundary Element Method (BEM) which promises to bring improved accuracy for the analysis of engineering problems, particularly those which involve infinite domains and high stress gradient.

In this paper, several techniques using both FEM and BEM are employed for the calculation of the Stress Intensity Factor K_I (Deformation Mode I). The results obtained lead to some practical conclusions which may be easily employed by engineers in the daily evaluation of structural integrity. The use or not of special quarter point elements is considered. FEM results are obtained using the ANSYS commercial software, which calculates automatically results for K_I using the J-Integral technique employing both common and quarter-point quadratic elements, the latter having a singularity. The BEM results are obtained using a classic plane stress FORTRAN code adapted for fracture mechanics problems to employ the COD (Crack Opening Displacement) method, Nodal Stresses and the J-Integral technique to calculate K_I. The code employs common quadratic elements, but the calculation of the derivatives of the displacements (u_1 and u_2, respectively, displacements for x and y directions) or ($\partial u_1/\partial x$ and $\partial u_2/\partial x$) which appear in the J-Integral is done in an accurate way by differentiating the BEM fundamental solutions.

2. Singular quarter - point elements in FEM and in BEM

The displacements and the stresses in BEM tend to infinity as a field point approaches a source point (Brebbia and Dominguez, 1989). This singularity has been the subject of much research in BEM and several techniques exist in order to integrate correctly the improper integrals which appear in fundamental solutions. On the other hand, the fundamental solutions used in BEM allow that even using only a few elements, the method is able to reproduce the high stress gradients which occur near a crack tip. In FEM the singularities are more difficult to model. Even so, both in FEM and in BEM the most accurate models of the singularities are obtained with the use of special elements in the region of the crack tip. Quadratic elements are usually preferred for 2D problems. The most popular singular elements for modelling crack tips, both in BEM and FEM are known as the quarter-point elements (Barsoum, 1976 and Blandford, 1981). These elements are obtained by moving the position of the midside node from /2 to the quarter-point position /4 and in FEM have lead to a considerable improvement in the modelling of the singularity.

It can be seen (Barsoum, 1976) that after moving the node from the mid-side to the quarter-point a term $1/\sqrt{Lr}$ has been introduced into the expression for strain in such a way that the strain (and as a consequence, the stress) approaches infinity as $r \rightarrow 0$. In FEM considering the displacement method, Ku=f, where K is a stiffness matrix, u is a vector of unknown displacements and f are applied forces. Once the displacements are obtained, the strains and then the stresses are calculated.

In BEM, the displacements (vector u) and the surface stresses (p) are calculated separately (Brebbia and Dominguez, 1989). For this reason in order to employ quarter point elements with BEM it is necessary to transform the shape functions for the surface stresses (p) to obtain the stress singularity at the crack tip (Blandford, 1981). Shifting the midside node to the quarter-point does not generate singularities in the displacements for which the shape functions are identical to those normally used in FEM for one element side, generating in this case an element known as the displacement quarter point (QPD) element. A possible representation of the stress singularity in BEM requires the multiplication of the shape function for the surface stresses (p) by the factor $\sqrt{L/r}$ (Blandford, 1981), generating in this case an element known as the traction quarter point (QPT) element.

In Fig. 1(a) the behaviour of common quadratic shape functions is shown. In Fig. 1(b) the behaviour of the shape functions for the quarter-point element for calculating stresses in BEM are shown for comparison.

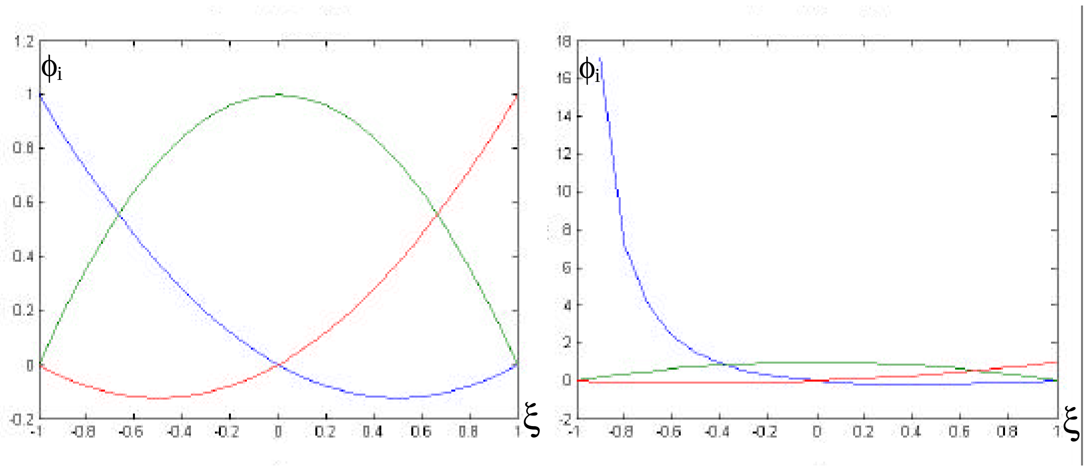


Figure 1. (a) Common quadratic shape functions without singularity. (b) Quadratic quarter – point shape functions, one of which contains a singularity, see Eq. (16)

Fig. 2 shows the stress results obtained for the case of the plate of width W=10 and with a crack size of a=l, giving a relationship a/W=0.1. The Boundary Element Method using three different types of elements for the discretization of the crack tip, i) the usual quadratic boundary element denominated QE in Fig. 2, ii) the displacement quarter point element QPD, and iii) the traction quarter point element QPT. The results obtained for other crack configurations and geometries are given in a similar way.

3. Revision of some methods for calculating k_i

Considering the LEFM literature, one of the methods for calculating the Stress Intensity Factor which is most popular amongst researchers, is the COD or Crack Opening Displacement method, (Blandford, 1981; Martínez and Domínguez, 1984). This method is used because of the simplicity of its implementation and because it allows the value of the stress intensity factor to be obtained directly. This methodology can be used to obtain K_I for different crack geometries, taking into account the displacements at one or more nodes employed in the crack discretization. Eq. (1) is written for non-symmetric crack (Banerjee, 1993).

$$K_I = \frac{2\mu}{\alpha+1}\sqrt{\frac{2\pi}{l}}\left[\left(4u_2^B - u_2^D\right) + u_2^E - u_2^C\right] \tag{1}$$

in the above u_2^j is the displacement in the direction y (in which the crack opens) at the point considered, $\alpha = (3-4\nu)$ for plane strain and $\alpha = (3-\nu)/(1+\nu)$ for plane stress and μ is the shear modulus. When the crack is symmetric, Fig. 3, the formula becomes

$$K_I = \frac{2\mu}{\alpha+1}\sqrt{\frac{2\pi}{l}}\left(4u_2^B - u_2^C\right) \tag{2}$$

As an alternative, if only one node is used for the calculation, the formula becomes

$$K_I = \frac{\mu}{\alpha+1}\sqrt{\frac{2\pi}{l}}\,u_2^B \tag{3}$$

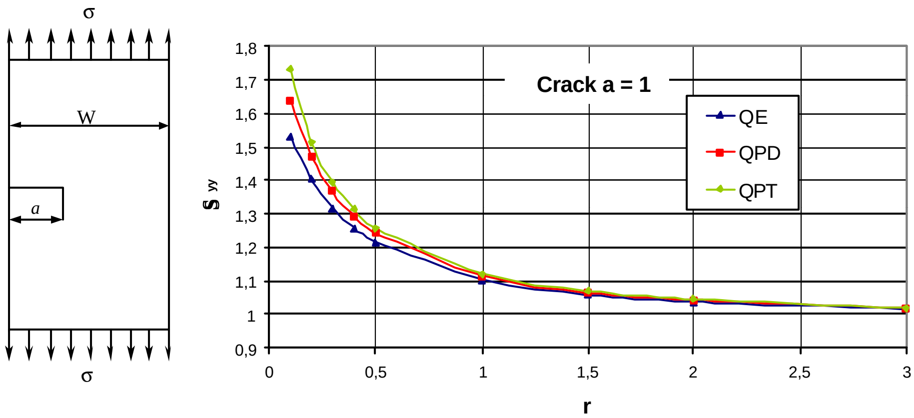


Figure 2. Stress results (σ_{yy}) for different types of boundary elements employed for the discretization of the crack tip

This method is preferred principally when singular quarter-point elements are employed. Fig. 3 shows a crack discretized in such a way as to be able to calculate K_I using Eq. (1). In order to use Eq. (2), the points D and E are not necessary. Another straightforward method for obtaining K_I is through the stress at point A employing the expression

$$K_I = p_2^A\sqrt{2\pi L} \tag{4}$$

Where p_2^A the stress σ_{yy} at the crack tip.

The use of the J-Integral for obtaining K_I is based on the evaluation of a line integral which follows an arbitrary path within the domain, which surrounds the crack tip, under the condition that the path both starts and finishes on the faces of the crack, as shown in Fig. 4. It was shown by Rice (1968), that the value of the following line integral J is independent of the path Γ. For 2D problems, this integral can be written as follows

$$J = \int_{\Gamma_j}\left(U dy - p\frac{\partial u}{\partial x}ds\right)d\Gamma \tag{5}$$

where U is the strain energy, dy is the variation in the y axe of the points used to calculate integral J, p is the vector of surface stresses on the plane of the normal n, external to the boundary used to compute the integral J, u is the displacement vector (for plane problems u = $\{u_1, u_2\}$ and $\partial u/\partial x = \{\partial u_1/\partial x$ and $\partial u_2/\partial x\}$ and ds is an infinitesimal element of the path of the line integral Γ (Ewalds and Wanhill, 1984). The values of the terms $\partial u/\partial x$ and $\partial v/\partial x$ can be determined in various ways. The simplest approach for calculating these values is the finite difference method. These values can also be obtained by implicit differentiation, which can produce the best results, and which was implemented in the BEM code used here. In the next section the methodology used for the calculation of these sensitivities using implicit differentiation is detailed.

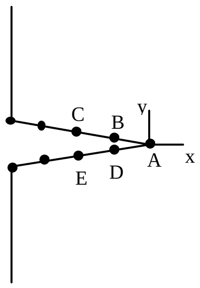


Figure 3. Calculation of K via COD (crack opening displacement)

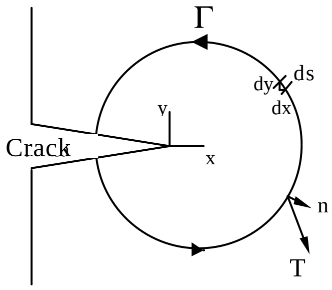


Figure 4. Path for the J-Integral: anti-clockwise sense, surrounding the crack tip

The value of the J-Integral is related directly to the stress intensity factor K (Broek, 1989). For deformation mode I,

$$K_I = \left(\frac{8\mu J}{1+\alpha} \right)^{1/2} \tag{6}$$

4. Implicit differentiation

Saigal and Kane (1989) presented a method for implicit differentiation of integral equations in order to determine the sensitivities of two-dimensional objects. To obtain these sensitivities it is necessary to differentiate the fundamental solutions used in BEM. The complete boundary solution is obtained from the classical Eq. (7) where H and G are the matrices with fundamental solutions and "u" and "p" are, respectively, displacement and traction (Brebbia and Dominguez, 1989). Using the known boundary values, internal values of u and p may be then calculated.

$$Hu = Gp \tag{7}$$

Note that in Eq.(7), u and p are values of displacements and boundary stresses respectively. Introducing the boundary conditions $\bar{u}$ and $\bar{p}$ on the boundary of the body and rearranging the terms in Eq. (7) in such a way that the unknowns are grouped together on the left in a vector w the algebraic system of equation

$$Aw = b \tag{8}$$

b is a known vector and A is a matrix with known coefficients after reordering of Eq.(7). To calculate the sensitivities it is necessary to differentiate Eq (7) with respect to x one obtains Eq.(9). Here $\bar{u}$ and $\bar{p}$ are known values of u and p.

$$(Hu)_{,x} = (Gp)_{,x} \text{ or } H_{,x}\bar{u} + H_{,x}u + H u_{,x} + H\bar{u}_{,x} = G_{,x}p + G_{,x}\bar{p} + G p_{,x} + G\bar{p}_{,x} \tag{9}$$

in the above, the terms in the matrices $H_{,x} = \left[H^{ij}_{,x}\right]$ and $G_{,x} = \left[G^{ij}_{,x}\right]$ are given by the derivatives of the fundamental solution in H and G matrices of Eq.(8) (Brebbia and Dominguez, 1989), as follows (Saigal and Kane, 1989).

$$H^{ij}_{,x} = \sum_t \int_{\Gamma_t} p^*_{,x}\Phi_q d\Gamma \quad \text{and} \quad G^{ij}_{,x} = \sum_t \int_{\Gamma_t} u^*_{,x}\Phi_q d\Gamma \tag{10}$$

where the derivatives of the fundamental solutions $p^*_{,x} = \left[p^*_{ij,x}\right]$ and $u^*_{,x} = \left[u^*_{ij,x}\right]$ are given by

$$u^{*}_{ij,x} = \frac{1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\delta_{lk}\frac{r_{,x}}{r} - \frac{1}{r^2}\left(y_{i,x}y_j + y_i y_{j,x}\right) + \frac{2}{r^3}y_i y_j r_{,x}\right] \quad (11)$$

$$p^{*}_{ij,x} = -\frac{1}{4\pi(1-\nu)r^2}\left[(1-2\nu)\left(n_j y_{i,x} + n_{j,x}y_i - n_i y_{j,x} - n_{i,x}y_j\right) - \left\{\frac{2y_i y_i}{2}\left(y_{k,x}n_k + y_k n_{k,x}\right)\right.\right. \quad (12)$$

$$\left.\left. + \frac{2}{r^2}\left(y_{i,x}y_j + y_i y_{j,x}\right)y_k n_k - \frac{4y_i y_j}{r^3}y_k n_k r_{,x}\right\}\right] + \frac{2}{4\pi(1-\nu)r^3}\left[(1-2\nu)\left(n_j y_i + n_i y_j\right) + \left((1-2\nu)\delta_{ij} + \frac{2y_i y_j}{r^2}\right)y_k n_k\right]r_{,x}$$

In the above $r_{,x} = (y_k y_{k,x})/r$ and $n_{i,x} = S^{-1}x_{i,\xi x} - S^{-2}S_{,x}x_{i,\xi}$ and x and y are the coordinates of the point, $n_{i,x}$ is the derivative of the normal in relation to x, S is the Jacobean and the remaining terms have been defined previously. Given that the derivatives of the known values are zero, Eq. (9) becomes finally

$$H\,u_{,x} - G\,\bar{p}_{,x} = G_{,x}\bar{p} + G_{,x}p - \bar{H}_{,x}\bar{u} - H_{,x}u \quad (13)$$

the values on the right hand side of Eq. (13) are all known, having been obtained in order to calculate the stresses at interior points. In this way, the system can be reordered as

$$Ay = d \quad (14)$$

The matrix A in the two systems of Eqs (8) and (14) is the same, thus reducing considerably the computational effort necessary to solve Eq. (14), the matrix obtained earlier for the calculation of boundary values of u and p may be stored and reused for calculating the sensitivities. The values of y obtained solving Eq. (14) are the sensibilities values on the path of the Integral - used for the calculation of the J-Integral. b is a known vector and A is as defined before.

5. Numerical examples

Panel with an edge crack: In this classical example a panel with an edge crack is considered, (Saouma, 2000). The geometry of the panel is defined by the parameters W=10cm, Fig. 6, and height 30cm. The size of the crack is considered to be a. Values of a of 1,2,3,4 and 5 cm are considered. The load shown in Fig. 6 is considered to be s_o=100kN/cm^2. The Young's modulus for the panel is considered to be 20500kN/cm^2 and the Poisson coefficient is 0.3. There is a closed analytic solution for this problem obtained by Keer & Freedman (1973), employing the expression for K_I given in Eq. (15) below with the constant "C" given in Eq. (16) which is valid for a ratio a/W up to 0.6

$$K_I = C\sigma_o\sqrt{\pi a} \quad (15)$$

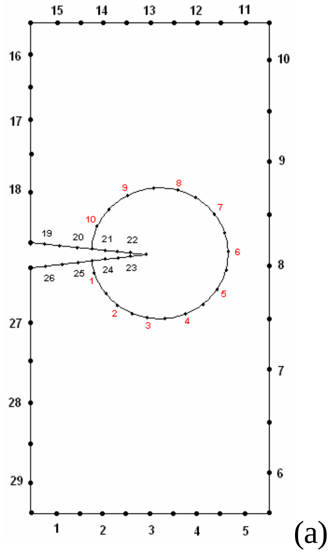
$$C = 1{,}12 - 0{,}231\left(\frac{a}{W}\right) + 10{,}55\left(\frac{a}{W}\right)^2 - 21{,}72\left(\frac{a}{W}\right)^3 + 30{,}39\left(\frac{a}{W}\right)^4 \quad (16)$$

In Fig. 9 the values are given for the error obtained in the calculation of K_I using FEM with the program ANSYS and using BEM with the FORTRAN code mentioned earlier and employing the J-Integral. For BEM two discretizations were considered, the first with 29 elements, 58 boundary nodes and 116 degrees of freedom, shown in Fig 5(a), the second with 58 elements, 116 boundary nodes and 236 degrees of freedom.

It is noted that results produced by the BEM code, even employing the discretizations considered, are similar to those produced by FEM using a mesh with 2000 elements, more than 4500 nodes and approximately 9500 degrees of freedom shown in Fig 5(b), which was generated automatically. For this problem, 5 different discretizations were considered, varying the size a of the crack, thus altering the ratio a/W. Using the special quarter-point elements the errors in the results obtained using ANSYS was reduced to 0.3%. The results obtained using the BEM code in which K is obtained using the J-Integral were of the order of 1.7% for the discretization shown in Fig. 5(a). With refinement, doubling the number of elements from 29 to 58 the error produced by the BEM code was reduced to approximately 0.5%. The results obtained using BEM and FEM are shown in Fig. 9 for comparison.

The plot shown in Fig. 10 gives the analytical value according to author Keer and Freedman (1973), that obtained employing the usual quadratic boundary element (QE), in this case repeating the values shown in Fig. 9, and those obtained using the Traction Quarter Point boundary element QPT. Implicit differentiation was used. Note that there exists reasonable agreement for K_I up to a/W=0.3, after this value the results obtained for K_I by the special element are larger, probably due to the larger stresses obtained using this element.

Panel with centre crack: In this example a panel with a centre crack is considered. The geometry and loading is shown in Fig. 7 (Saouma, 2000). For this problem there exists an analytical solution, obtained by Paris & Sih (1965). K_I can be calculated using Eq. (15) however the constant "C" is obtained using the following Eq. (17).



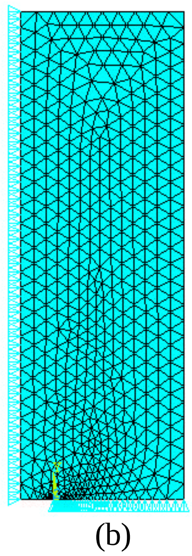


Figure 5. (a) BEM discretization employing 29 boundary elements for the complete problem (b) FEM mesh used by ANSYS code for discretizing the same problem taking advantage of symmetry

$$C = 1 + 0{,}256\left(\frac{a}{W}\right) - 1{,}152\left(\frac{a}{W}\right)^2 + 12{,}200\left(\frac{a}{W}\right)^3 \tag{17}$$

Fig. 11 shows the values of the error obtained in the calculation of K_I using FEM with the ANSYS code and using BEM with the code employing the J-Integral. The agreement between the results obtained in both cases is good. The discretizations employed for FEM and BEM are the same as those employed in the previous example. For this case only half of the problem is considered due to symmetry and the boundary conditions are specified in accordance with Fig 9.

The plot shown in Fig. 12 shows the analytic values obtained by Paris and Sih (1965), those obtained for the usual quadratic boundary element, denominated QE above, which are the same as those given in Fig. 11, and the results obtained by the Traction Quarter Point boundary element, QPT. The results were obtained using implicit differentiation. Note that there exists reasonable agreement in the values obtained for K_I up to a/W = 0.3 but after this, the results obtained for K_I by the special element are higher, probably due to the fact that the results for stresses obtained by this element are higher.

Panel with two edge cracks: In this example a panel with a crack in two edges is considered with the geometry and loading shown in Fig. 8 (Saouma, 2000). For this problem an analytical solution has been obtained, (Paris e Sih, 1965). The discretizations employed for FEM and BEM are once again the same as in the previous example, advantage is taken of symmetry, and the boundary conditions are specified as in Fig. 5. K_I is given by Eq. (15) with the constant "C" obtained using the following equation

$$C = \left[1{,}12 - 0{,}561\left(\frac{a}{W}\right) - 0{,}205\left(\frac{a}{W}\right)^2 + 0{,}471\left(\frac{a}{W}\right)^3 - 0{,}190\left(\frac{a}{W}\right)^4\right] \Big/ \left[\sqrt{1 - \frac{a}{W}}\right] \tag{18}$$

Fig. 13 shows the values of the error obtained in the calculation of K_I using FEM employing the ANSYS code and using the BEM employing the J-Integral. Results obtained for this case with the Traction Quarter Point element are shown in Fig. 14 and are similar to those shown in Fig. 12 for the case of the centered crack

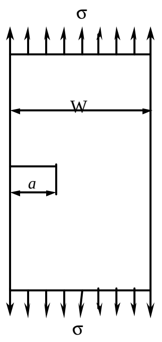


Figure 6. Panel with edge crack with of length "a"

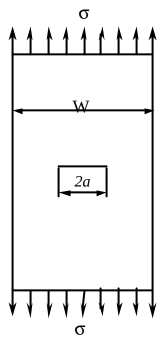


Figure 7. Panel with center crack

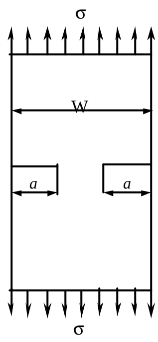


Figure 8. Panel with two edge of length "a"

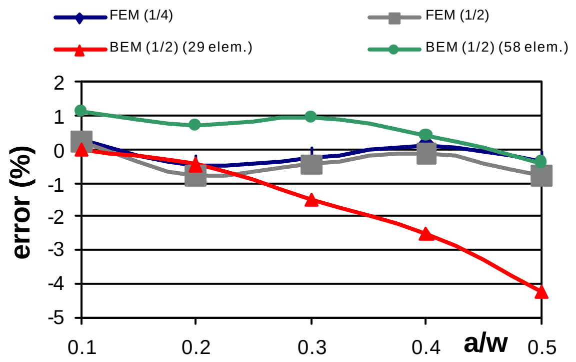


Figure 9. Error in results obtained for K_I for panel edge crack using BEM and FEM

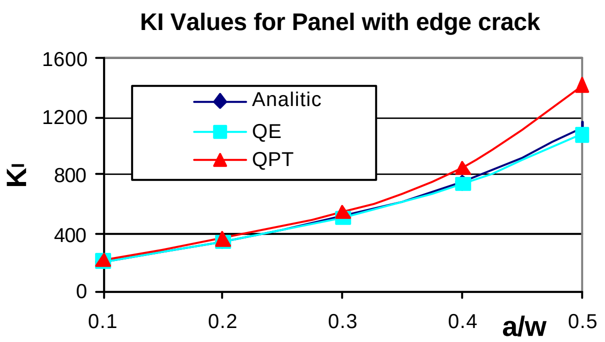


Figure 10. Results obtained for KI for edge crack

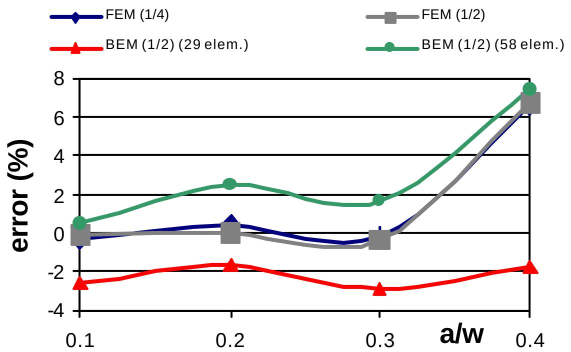


Figure 11. Results for error in KI for panel of length 2a - central

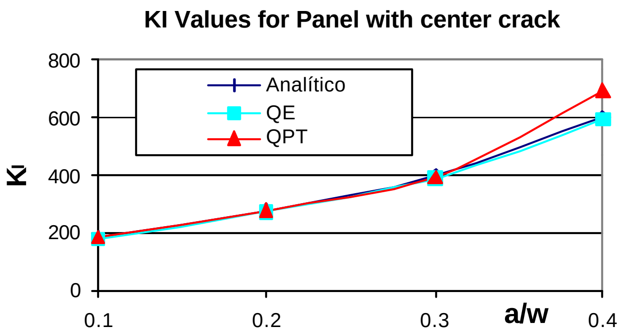


Figure 12. Results obtained for KI for center crack

6. Conclusions

The results obtained here show that accurate values of the stress intensity factor can be obtained using numerical methods. The accuracy of the results improves when quarter point elements are employed and also if finer discretizations are used. Comparing the results obtained here with those given in the literature, it is noted that the simple methods such as COD and the Nodal Stress method can give reasonable values of K when refined meshes and quarter-point elements are employed with FEM. The technique which relates the value of K to the J-Integral requires a larger computational effort, but produces much more accurate results. In FEM, the results obtained using the J-Integral and employing quarter-point elements contained only small errors in comparison with solutions available in the literature. The formulation presented for BEM also gave results compatible with the best results obtained using FEM even considering that the BEM code employed did not use the special quarter-point elements.

BEM produced good results for discretizations employing only a few elements using implicit differentiation of the fundamental solutions for the calculation of $\partial u_1 / \partial x$ and $\partial u_2 / \partial x$ in the calculation of the J-Integral. Considering that the singularity in LEFM occurs in a limited region and is a mathematical artifice not present in real materials, and considering that the stress distribution (with or without the singularity) is in equilibrium with the applied loading, and given the results obtained using BEM without the use of the singular elements, one can ask of the search for the singularity at the crack tip in numerical methods is essential for the calculation of K via the J-Integral. The results obtained with the use of the quarter point elements for the stress intensity factor are slightly higher than those obtained without quarter point up to a/W = 0.3, after this the results obtained by the special element are considerably higher.

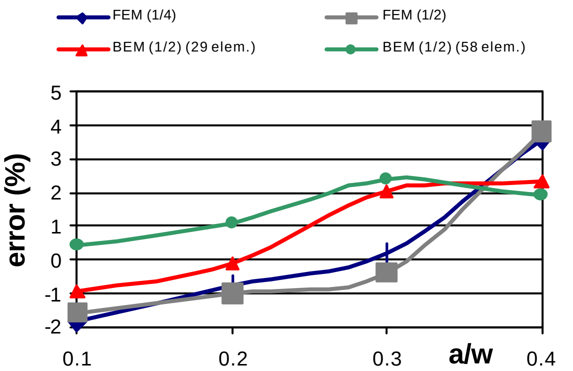


Figure 13. Results for error in K_I for panel cracks with two edge cracks

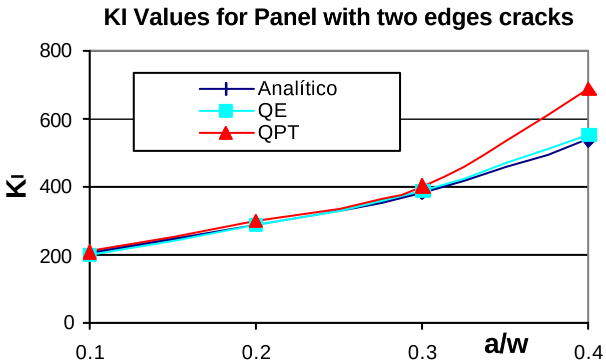


Figure 14. Results for K_I for two edge cracks

7. Acknowledgements

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